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ABSTRACT

This manual outlines an advanced mathematics course for secondary school students in Manitoba (Canada). Included are eight different topics for a total of ten modules, with each module accounting for 18 40-minute periods. Teachers, or teachers and students cooperatively, should select the equivalent of five modules from the eight topics covered. The course is designed to be of interest to highly motivated students who wish some introduction to areas of mathematics which will be studied in depth in post-secondary programs. Six general objectives are listed. Included are an introduction, examples, objectives, and notes to teachers. The topics covered are: complex numbers and polar coordinates; statistics; number theory; linear programming; theory of equations; matrices and systems of linear equations; three-dimensional geometry; and vectors. Appendices contain annotations of primary and supplementary references. (MNS)

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1987

advanced mathematics 305
optional half credit

interim guide

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advanced mathematics 305
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THIS GUIDE IS SUBJECT TO REVISION ON THE BASIS OF INFORMATION RECEIVED FROM SCHOOLS. IT IS EXPECTED THAT MODIFICATIONS WILL NOT BE IN THE DESIGN OF THE PROGRAM. RATHER, CHANGES MAY BE MADE IN AREAS CONCERNING STRATEGIES, RESOURCE MATERIALS, AND IMPLEMENTATION.

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overview

OVERVIEW

INTRODUCTION

Advanced Mathematics 305 is not to be substituted for Mathematics 300. A student enrolled in Advanced Mathematics 305 should have completed or be currently enrolled in Mathematics 300.

A module in this guide accounts for a total of eighteen, 40-minute periods. This allows for 14 periods for new work and four periods for review and testing. Five modules constitute a complete half-credit course.

The program outlined in the following pages includes eight different topics for a total of ten modules. Teachers or teachers and students cooperatively should select the equivalent of five modules from the eight topics covered.

Note: Topic II - Statistics should be selected if students have not taken "Probability and Statistics 305" but should not be selected if students have already taken or are currently taking "Probability and Statistics 305."

RATIONALE

Three sequences of mathematics courses are offered, in grades 10-12, to meet the needs of the majority of students.

The 04 program is designed to concentrate on the application of mathematical skills and concepts to everyday living and is a terminal course. The 01 program deals with applications to business, trades, and vocational areas and leads to a range of non-mathematics oriented post-secondary training programs. The 00 program is a theoretical program of algebra, geometry and trigonometry with some consumer-related topics, leading to post-secondary training in mathematics and science-oriented courses.

The optional half-credit courses, Statistics and Probability, Advanced Mathematics and Introduction to Calculus, are designed to meet the needs of students who demonstrate particular aptitude in mathematics and who desire to study more advanced topics. Increasingly, students are demanding extra preparation in mathematics to enable them to adapt more readily to university courses.

An understanding of elementary probability and statistical theory is advantageous to students entering into business administration, science, and psychology, as well as to those planning to specialize in mathematics.

The various topics in Advanced Mathematics and Introduction to Calculus expand the students' experience of mathematics by introducing applications of the theory of the 00 programs and also some recent developments in mathematics.

The three optional half-credit courses are of interest to highly motivated students who wish some introduction to areas of mathematics which will be studied in depth in post-secondary programs. These courses are of particular interest to those students planning more advanced studies in Engineering, Math-

ematics, Science, and other mathematics-oriented faculties at out-of-province universities where it is expected that entering students will have some knowledge of the content of these optional half-credit courses.

The inclusion of these optional half-credit courses will complete the range of departmental programs which will meet the needs of students at all ability and interest levels.

USE OF MATHEMATICS

It is important to study mathematics because of its many applications to everyday life and to specialized scientific work. A knowledge of mathematics is necessary if students are to have an awareness and an understanding of their environment and to enjoy a fuller life.

With this in mind, it is suggested that mathematics be considered:

1. A mode of communications with a language all its own. For students to communicate successfully in mathematics, they must learn the basic concepts without having to meet demands for logical rigour. The language acquired should be used to describe relationships observed in the real world. At a later stage, symbols should be introduced to record these relationships.

2. A tool. After a mathematical system has been introduced as a language, it should be used as a tool to answer practical questions, thus demonstrating to students that to function in society they need basic skills in arithmetic, algebra, and statistics. Students should be capable of identifying a problem and employing an organized process leading to its solution. This involves applying fundamental operations and information processing skills--such as collecting, organizing and interpreting data--to familiar and unfamiliar contexts. Realistic problems relating to the students' environment should be sought to foster consumer awareness, and vocational and business applications should be stressed at every opportunity.

3. A source of esthetic enjoyment. Mathematics can be considered an esthetic discipline like music. Many students can develop an appreciation of mathematics through the discovery of patterns in numbers and in created or naturally occuring designs. Students should be encouraged to reproduce these as well as to create their own. Methodologies and techniques conducive to developing positive inquiring attitudes toward mathematics should be employed.

   Students who are most successful in developing skills and understanding concepts in mathematics have learned to recognize and appreciate the three ways in which the subject can be used. Classroom emphasis on these uses cannot always be equal, however, but must vary depending on the nature of the topic and the interests and abilities of the students.

GOALS OF THE SENIOR HIGH SCHOOL MATHEMATICS PROGRAM

General Goal of Mathematics

The aim of mathematics is to foster maximum and continuous growth in the following behavioral areas: the cognitive (factual), the affective (feelings,

perceptions), and the psychomotor (movements). This growth will be accomplished when mathematical content is used to develop processes such as the following:

1. The process of communicating: using mathematical language and symbolism.
2. The process of recalling mathematical skills.
3. The process of comprehending mathematical principles.
4. The process of seeing relationships in mathematics.
5. The process of generalizing: finding common relationships among mathematical concepts.
6. The process of making applications: applying generalizations to new situations.
7. The process of learning self-direction: searching for mathematical ideas.
8. The process of discovering: recognizing mathematical ideas.
9. The process of creating: generating mathematical ideas.
10. The process of appreciating the place of mathematics: understanding its widespread applications to our daily activities.
11. The process of problem solving: identifying, hypothesizing, interpreting, relating, evaluating, concluding.

General Objectives of Advanced Mathematics 305

1. To apply and further develop some of the topics introduced in Mathematics 100, 200 and 300.
2. To develop an awareness of the overall development or the overall scope of mathematics.
3. To prepare the student for further courses in mathematics and related subject areas.
4. To emphasize algebraic manipulation and the development of sound problem solving techniques.
5. To use computers whenever efficient to do so to enhance the understanding of mathematics concepts and principles.
6. To demonstrate many practical applications of mathematics.

REFERENCES

Primary References

Since no single text covers the complete basic outline, it is recommended that each teacher have a copy of some of the texts listed below. It is not suggested that class sets be purchased.

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Shanks, M.E. Pre-Calculus Mathematics. 2nd ed. Don Mills, ON: Addison-Wesley Publishing, 1972.

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Nichols, D. Modern Intermediate Algebra. Rev. ed. Toronto, ON: Holt, Rinehart and Winston, 1970.

Protter, M.H. and Morrey, C.B. Calculus with Analytic Geometry: A First Course. Don Mills, ON: Addison-Wesley, 1977.

Vance, E.P. An Introduction to Modern Mathematics. Don Mills, ON: Addison-Wesley, 1968.

Note: Several of the primary and supplementary references are annotated in Appendices I and II.

topics

TOPIC I: COMPLEX NUMBERS AND POLAR COORDINATES (2 MODULES)

INTRODUCTION

Teachers may wish to introduce this topic by illustrating the need for numbers beyond the real numbers. Consider, for example, a solution to an equation such as $x^2 = -9$. Explain that while complex numbers have limited application in regular daily use, they have wide-spread applications in many branches of physical science and engineering, where the real-number system is not adequate to describe the physical processes.

Any electrical circuit in which an alternating current is flowing cannot be analysed fully without the use of complex number notation. (Note: Ordinary house current is alternating current and nearly all hydro power lines carry alternating current.)

The whole field of telecommunication design (AM, FM, CB) depends on the use of complex numbers. Similarly, control-systems analysis (e.g., airplane automatic pilot or automobile power-steering or missile guidance systems) requires in-depth knowledge of complex numbers.

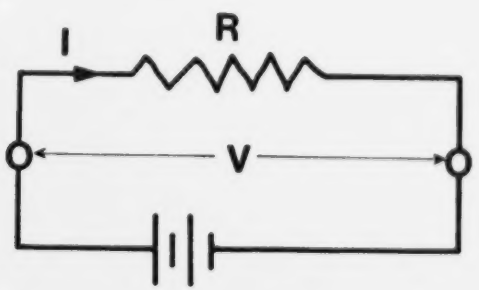
Many more examples can be found, not only in electrical engineering but in mechanical, civil, and aeronautical engineering.

Example 1:

The equation V=IR is used to analyze electrical circuits in which a direct current is flowing.

V = Voltage in Volts
I = Current in Amperes
R = Resistance in Ohms

'V' is the voltage to be measured across the resistance 'R'.

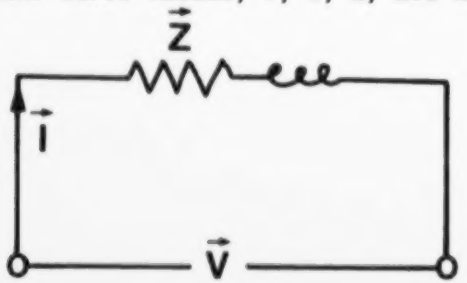


Example 2:

The equation is modified to $\vec{V} = \vec{I}\vec{Z}$ for circuits in which an alternating current flows.

$\vec{V}$ = Voltage in Volts
$\vec{I}$ = Current in Amperes
$\vec{Z}$ = Impedance in Ohms

Note: All three values, $\vec{V}$, $\vec{I}$, $\vec{Z}$, are now complex quantities.



If the current flowing is $\vec{I} = (3 + 4i)$ amperes and the impedance is $\vec{Z} = (2 + 2i)$ Ohms, determine:

a) The modulus of $\vec{I}$; i.e., $(\vec{I})$,
b) The modulus of $\vec{Z}$; i.e., $(\vec{Z})$,
c) The modulus of $\vec{V}$; i.e., $(\vec{V})$.

Note: Any of the quantities, $\vec{V}$, $\vec{I}$ or $\vec{Z}$, can contain a negative factor. For example, $\vec{I} = (-4 + 6i)$ amperes is possible.

$$|\vec{I}| = \sqrt{9 + 16} = 5$$
$$|\vec{Z}| = \sqrt{4 + 4} = \sqrt{8}$$
$$|\vec{V}| = (3 + 4i)(2 + 2i)$$
$$= 6 + 6i + 8i - 8$$
$$= -2 + 14i$$
$$|\vec{V}| = \sqrt{4 + 196}$$
$$= \sqrt{200}$$

Before beginning this topic, review the development of the real-number system. The following sketch shows the development of the real number system:

natural ⟶ whole ⟶ integer ⟶ rational ⟶ real
integer ⟶ irrational ⟶ real

OBJECTIVES	NOTES TO TEACHERS

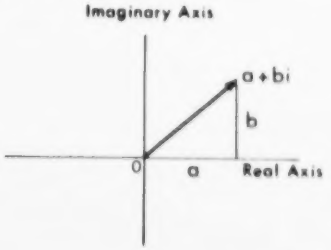
The student should be able:

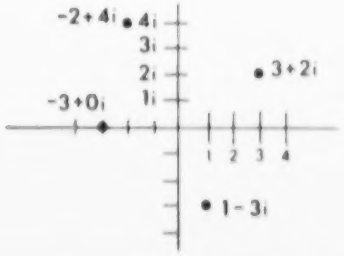
1. to define complex numbers;

A complex number is any number that can be represented in the form a + bi where a and b are real numbers and $i = \sqrt{-1}$. a is referred to as the real part and b as the imaginary part of a + bi. By agreement, $0 \cdot i = 0$, therefore, every real number is a complex number since $a = a + 0 = a + 0i$. Generally, a complex number is represented by the letter z.

2. to represent complex numbers geometrically;

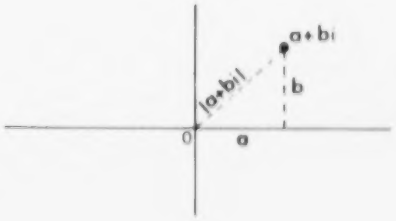
Any real number can be represented as a point on a line. Since a complex number a + bi is determined by two real numbers a and b, i.e., an ordered pair, (a,b), (a,b) determine a point on a plane. Such a plane is called an Argand or complex plane.





3. to determine the absolute value of a complex number;

The absolute value of a real number a is the distance from the origin on a real number line. Similarly, the absolute value of a complex number, a+bi , is the distance from the origin on the argand plane. Therefore, $a+bi = \sqrt{a^2+b^2}$.



OBJECTIVES	NOTES TO TEACHERS

Examples:

1. Find the absolute value of $3 + 4i$.

   Solution:

   $$|3 + 4i| = \sqrt{3^2 + 4^2}$$

   $$= \sqrt{25}$$

   $$= 5$$

2. Find the absolute value of $- - 6i$.

   Solution:

   $$|0 - 6i| = \sqrt{0^2 + (-6)^2}$$

   $$= \sqrt{36}$$

   $$= 6$$

4. to add and subtract complex numbers;

Addition: $(a+bi) + (c+di) = (a+c) + (b+d)i$

Subtraction: $(a+bi) - (c+di) = (a+bi) + (-c-di)$

$= (a-c) + (b-d)i$

Examples:

1. $(3+2i) + (-5+4i) = (3-5) + (2+4)i$

$= -2 + 6i$

OBJECTIVES	NOTES TO TEACHERS
	2. $(6-3i) - (4+5i) = (6-4) + (-3-5)i$ $= 2 + (-8i)$ $= 2 - 8i$
5. to multiply complex numbers;	$(a+bi) \times (c+di) = (ac-bd) + (ad+bc)i$ The multiplication of complex numbers can be carried out by the same method that polynomials are multiplied. It must be remembered that i^2 can be replaced by -1. Example: $(2-3i) \times (4-i) = 8 - 2i - 12i + 3i^2$ $= 8 - 14i + 3(-1)$ $= 5 - 14i$
6. to define a complex conjugate;	a-bi is the complex conjugate of a+bi. The product of a complex number and its conjugate is a real number. Symbolically, if z is a complex number, then $\bar{z}$ represents its conjugate. Example: $(5+2i) \times (5-2i) = 25 - 4i^2$ $= 25 + 4$ $= 29$

OBJECTIVES	NOTES TO TEACHERS
7. to divide complex numbers;	Multiply the numerator and denominator by the conjugate at the denominator. Example: $\frac{3 - 2i}{5 + 3i} = \frac{3 - 2i}{5 + 3i} \times \frac{5 - 3i}{5 - 3i}$ $= \frac{9 - 19i}{34}$ $= \frac{9}{34} - \frac{19}{34}i$
8. to prove certain properties of complex numbers;	1. If z_1 and z_2 represent complex numbers, then $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ 2. If z_1 and z_2 represent complex number, then $\overline{(z_1)(z_2)} = (\overline{z_1})\,(\overline{z_2})$

OBJECTIVES	NOTES TO TEACHERS
9. to solve quadratic equations with real coefficients and having complex roots;	Example: Find the solution set of the following equation. $3x^2 - 4x + 10 = 0$ $x = \frac{4 \pm \sqrt{16-120}}{6}$ $= \frac{4 \pm \sqrt{104}}{6}$ $= \frac{4 \pm 2i\sqrt{26}}{6}$ $= \frac{2}{3} + \frac{i\sqrt{26}}{3}, \frac{2}{3} - \frac{i\sqrt{26}}{3}$ Note: After students have completed several examples, point out that complex roots always come in pairs.
10. to solve quadratic equations with complex coefficients;	Restrict examples to those which meet the following conditions: Either: a) a, b and c are imaginary b) a and c only are imaginary and b is real c) b only is imaginary and a and c are real The conditions stated above are necessary to avoid examples where the square root of a complex number must be calculated. This will be done later in objective 18.

Examples:

1. $x^2 + 3ix - 4 = 0$

Solution

$$x = \frac{-3i \pm \sqrt{9i^2 + 16}}{2}$$

$$= \frac{-3i \pm \sqrt{7}}{2}$$

$$= \frac{\sqrt{7}}{2} - \frac{3i}{2}, \frac{-\sqrt{7}}{2} - \frac{3i}{2}$$

2. $5ix^2 - 2ix + 3i = 0$

Solution:

$$x = \frac{2i \pm \sqrt{4i^2 - 60i^2}}{10i}$$

$$= \frac{i \pm \sqrt{14}}{5i}$$

$$= \frac{1}{5} + \frac{i\sqrt{14}}{5}, \frac{1}{5} - \frac{i\sqrt{14}}{5}$$

Alternate Solution

$i(5x^2 - 2x + 3) = 0$

$\therefore 5x^2 - 2x + 3 = 0$

$$x = \frac{2 \pm \sqrt{4 - 60}}{10}$$

$$= \frac{1 \pm \sqrt{-14}}{5}$$

$$= \frac{1}{5} + \frac{i\sqrt{14}}{5}, \frac{1}{5} - \frac{i\sqrt{14}}{5}$$

OBJECTIVES	NOTES TO TEACHERS

3. $2ix^2 - 6x + 3i = 0$

Solution

$$x = \frac{6 \pm \sqrt{36 - 24i^2}}{4i}$$

$$= \frac{3 \pm \sqrt{15}}{2i}$$

$$= \left(\frac{-3 + \sqrt{15}}{2}\right)i, \left(\frac{-3 - \sqrt{15}}{2}\right)$$

11. to form equations, given their complex roots;

Examples:

1. roots: $2 \pm i$

equation: $[x - (2 + i)][x - (2 - i)] = 0$

$[(x - 2) - i][(x - 2) + i] = 0$

$(x - 2)^2 - i^2 = 0$

$x^2 + 4x + 5 = 0$

Note: When the roots are a conjugate pair, the resulting equation has real coefficients.

2. roots: $3 + i, -2 - i$

equation: $[x - (3 + i)][x - (-2 - i)] = 0$

$(x - 3 - i)(x + 2 + i) = 0$

$x^2 - x - (5 + 5i) = 0$

Note: When the roots are not a conjugate pair, the resulting equation must have one imaginary coefficient.

OBJECTIVES	NOTES TO TEACHERS

3. roots: $(2 \pm i), (-1)$

equation: $[x - (2 + i)][x - (2 - i)][x + 1] = 0$

$(x^2 - 4x + 5)(x + 1) = 0$

$x^3 - 3x^2 + x + 5 = 0$

(cubic equation)

12. to plot points on a polar coordinate system;

Students should become familiar with the following terms: pole, polar axis, angle of rotation, polar coordinates. Examples should include use of both degree and radian measure. Students should be aware of various ways of describing the same point on a polar coordinate system.

Example:

The point (3, 30°) can be expressed as (3, 390°), (3, -330°), (-3, 210°), etc.

13. to convert rectangular coordinates to polar coordinates and vice-versa;

The following relations between rectangular and polar coordinates can be used:

$x = r \cos \theta$
$y = r \sin \theta$
$r = \sqrt{x^2 + y^2}$

$\tan \theta = \frac{y}{x}$

14. to convert equations in polar form to rectangular form and vice-versa;

Examples:

1. Change the equation $x^2 + y^2 = 2y$ to polar form.

OBJECTIVES	NOTES TO TEACHERS

Solution:

since $r^2 = x^2 + y^2$ and $y = r \sin \theta$
$x^2 + y^2 = 2y$ becomes
$r^2 = 2r \sin \theta$
or $r = 2 \sin \theta$

2. Change the equation $r^2 \cos \theta \sin \theta = 1$ to rectangular form.

Solution:

since $r^2 = x^2 + y^2$

$\cos \theta = \frac{x}{r}$

and $\sin \theta = \frac{y}{r}$

the equation becomes

$(x^2 + y^2)\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) = 1$

or $(x^2 + y^2)\frac{(xy)}{(x^2 + y^2)} = 1$

or $xy = 1$

Alternate Solution

$r^2 \cos \theta \cdot \sin \theta = 1$

$r \cos \theta \cdot r \sin \theta = 1$

$x \cdot y = 1$

$xy = 1$

OBJECTIVES	NOTES TO TEACHERS
15. to graph equations in polar form on a polar coordinate system;	Encourage students to use the symmetry of trigonometric functions to speed up the graphing process. The use of microcomputer to investigate a variety of popular equations would be appropriate at this point. The following types should be covered: line, circle, cardiod, three-petalled rose, parabola, ellipse, hyperbola.
16. to express a complex number in polar form;	To change a + bi to polar form, the relationships $r = a^2 + b^2$ and $\cos \theta = \frac{a}{r}$ are used. Example: For $3 - 4i$, $r = 5$ and $\cos \theta = \frac{3}{5}$, since $3 - 4i$ is in quadrant IV, $\theta = 323°$. Thus, $3 - 4i = 5 (\cos 323° + i \sin 323°)$. Note: $5(\cos 323° + i \sin 323°)$ can be expressed in a shorthand form; i.e., 5 cis 323°
17. to convert a complex number in polar to rectangular form;	Example: $3(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}) = 3 (\frac{1}{2} + \frac{\sqrt{3}}{2} i)$ $= \frac{3}{2} + \frac{3\sqrt{3}}{2} i$
18. to multiply and divide complex numbers in polar form;	Multiplication: $r(\cos \alpha + i \sin \alpha) \times s(\cos \beta + i \sin \beta) = rs[\cos (\alpha + \beta) + i \sin (\alpha + \beta)]$ Division: $r(\cos \alpha + i \sin \alpha) \div s(\cos \beta + i \sin \beta) = \frac{r}{s} [\cos (\alpha - \beta) + i \sin (\alpha - \beta)]$

OBJECTIVES	NOTES TO TEACHERS

19. to compute powers and roots of complex numbers using De Moivre's theorem.

If the students previously have studied double angle and half-angle trigonometry, show them the proofs of these methods.

De Moivre's theorem:

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Examples:

1. Evaluate and express in rectangular form $[3(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})]^5$.

Solution:

$$[3(\cos\tfrac{2\pi}{5} + i\sin\tfrac{2\pi}{5})]^5 = 3^5(\cos 2\pi + i\sin 2\pi)$$
$$= 3^5(1 + 0i)$$
$$= 3^5$$

2. Evaluate and express in rectangular form the fourth roots of $-8 + 8\sqrt{3}i$.

Solution:

Convert $-8 + 8\sqrt{3}i$ to polar form.
$-8 + 8\sqrt{3}i = 16[\cos(120 + 360k)^\circ + i\sin(120 + 360k^\circ)]$
where $k = 0, 1, 2, 3$

The fourth roots of $-8 + 8\sqrt{3}i$ =

$$16^{\frac{1}{4}}[\cos\frac{120 + 360}{4}k^\circ + i\sin\frac{120 + 360}{4}k^\circ]$$

OBJECTIVES	NOTES TO TEACHERS

1st root = $2[\cos 30° + i \sin 30°]$ $(k = 0)$

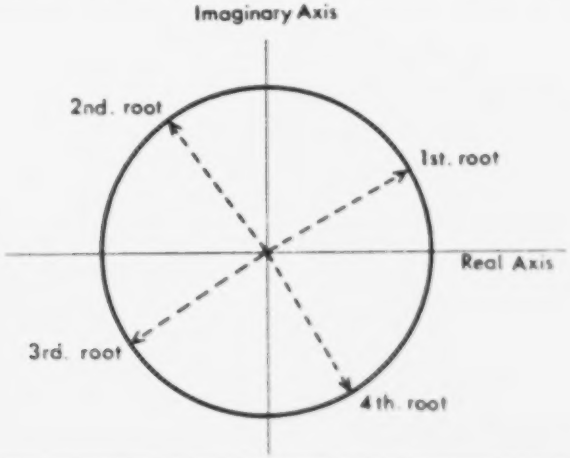
2nd root = $2[\cos 120° + i \sin 120°]$ $(k = 1)$

3rd root = $2[\cos 210° + i \sin 210°]$ $(k = 2)$

4th root = $2[\cos 300° + i \sin 300°]$ $(k = 3)$

$\therefore$ 4th roots of $-8 + 8\sqrt{3}i$ are $\sqrt{3} + i$, $-1 + \sqrt{3}i$, $-\sqrt{3} - i$, $1 - \sqrt{3}i$

The four distinct roots of $-8 + 8\sqrt{3}\,i$ may be plotted on the Argand plane. The points representing these fourth roots lie on a circle about the origin of radius 2. These points divide the circle into 4 equal parts.



ENRICHMENT

Teachers may wish to guide students in solving quadratic equations with complex coefficients at this time.

TOPIC II: STATISTICS (1 MODULE)

Statistics is the study of the collecting and analyzing of data from which inferences and conclusions can be made. The field of statistics is changing rapidly with technology. If computers are available in the school, they should be incorporated into this topic. Some calculators are equipped with a statistics mode. If they are available, their use should be incorporated into this topic. Some sets of data are given at the end of this topic to be used as required. Teachers may wish to refer to Jacobs, H. Mathematics: A Human Endeavour, W. H. Freeman and Company, San Francisco, 1970.

OBJECTIVES	NOTES TO TEACHERS
A. AREAS OF APPLICATION—USES OF STATISTICS	
The student should be able:	
1. to identify areas in which statistics are used.	Examples of some topics for discussion are: . mortality tables . insurance premiums . population movements . school construction and size . pension funds . hunting restrictions . evaluation of seat-belt laws
B. DATA GATHERING	
The student should be able:	
1. to distinguish between a census and a sample;	In a census, all elements of the population are polled. In a sample, only a representative percentage of the population is surveyed.
2. to list various methods of obtaining data;	Data may be obtained through interviews (telephone, mail-in, door-to-door survey), by measuring (velocity of cars from 0-10 seconds), or by counting (number of absences from school on Fridays).

OBJECTIVES	NOTES TO TEACHERS

3. to recommend methods of obtaining a representative sample.

The elements in a sample must have been chosen at random. Every element in the population must have an equal chance of being selected.

C. UNGROUPED OR RAW DATA

The student should be able:

1. to calculate from ungrouped data measures of central tendency;

a) mode,

The number which occurs most frequently in a set of data is called the mode.

Data: 3, 2, 6, 3, 7, 12, 3, 14

Solution: The mode is 3

b) mean,

The arithmetic mean $\bar{x}$, (bar x) is obtained by totalling the measures of a set and then dividing the sum by the number of measures in the set.

Example: Find the mean of 78, 82, 83, 79 and 23

Solution: $$\bar{x} = \frac{78 + 82 + 83 + 79 + 23}{5}$$

$$= 69$$

c) median;

If the data is arranged in order of magnitude, the middle measure is the median. If the number of the measures in a set is even, then the median is the mean of the two middle measures.

OBJECTIVES	NOTES TO TEACHERS

Example: 23, 78, 79, 82, 83. The median is 79

Example: 2, 6, 7, 8, 15, 24. The median is $\frac{7 + 8}{2} = 7.5$

2. to select the measure of central tendency most representative of the group of data;

Example:

Find the mean, median, and mode of the following:

2, 8, 4, 9, 6, 5, 4, 3, 2, 8, 9, 3, 7, 9

Mean = 5.6 Median = 5.5 Mode = 9

Which is the best measure of central tendency?

Each kind of measure is appropriate for certain types of data. One very high or very low number can distort the mean. The median is not affected by an extremely high or low number but may not be representative. For example, a group of 5 grandmothers and 4 infants might have a median age of 74, a mean of 42 and a mode of 80.

In a normal distribution, the mean, mode and median would be identical. Most things, however, are not statistically normal.

Examples:

If a man lays with his head in an oven (40°C) and with his feet in an ice bucket (0°C), could we say that on the average (mean) he is quite comfortable?

Did you know that the employees at Mr. Midas Skinflint's custom auto body earn an "average" of $20,000 a year? What you may not know is that each of his 3 employees has a salary of $7000 while he pays himself $59 000 a year.

OBJECTIVES	NOTES TO TEACHERS
3. to calculate measures of dispersion of a set of ungrouped data:	In statistics, the most common measures of dispersion or variability are range, mean deviation, variance and standard deviation. We shall consider only range and standard deviation. At times, measures of central tendency have little meaning. The length of an airport runway cannot be built to accommodate the "average" aircraft; a crane cannot be built to lift the "average" load. In such cases it would be better to consider the extremes or the dispersion.
a) range,	The range is a measure of the difference between the largest and smallest elements of a set. Example: Determine the range of the following data: 23, 46, 78, 12, 2, 9, 54, 97, 84, 13 Solution: Range 97 - 2 = 95
b) standard deviation.	The standard deviation is the square root of the average of the squares of the differences of these measures from the mean. Standard deviation = $\sqrt{\frac{\sum(x - \bar{x})^2}{N}}$, where x = a measure $\bar{x}$ = the mean for the set N = the number of measures

Example 1:

The number of hours per day that Jane practised playing the piano last week was 3, 2, 5, 4, 6, 6, 2. Find the standard deviation.

$$\text{Mean} = \bar{x} = \frac{3 + 2 + 5 + 4 + 6 + 6 + 2}{7} = \frac{28}{7} = 4$$

Number of Hours (x)	Difference from Mean ($x - \bar{x}$)	Square of the Difference $(x - \bar{x})^2$
3	-1	1
2	-2	4
5	1	1
4	0	0
6	2	4
6	2	4
2	-2	4
28		18

$$\text{Standard deviation} = \sqrt{\frac{18}{7}} \doteq \sqrt{2.57} \doteq 1.6$$

Example 2

An appliance store received a shipment of 4 refrigerators. The mass of each (in kg) is: 230, 190, 200, 220. Determine the standard deviation for these measures.

$$\bar{x} = \frac{230 + 190 + 200 + 220}{4} = \frac{840}{4} = 210$$

Mass x	Deviation $(x - \bar{x})$	Square of Deviation $(x - \bar{x})^2$
230	+20	400
190	-20	400
200	-10	100
220	+10	100
		1000

Standard deviation $= \sqrt{\frac{1000}{4}}$

$= \sqrt{250}$

$= 15.81$

D. GROUPED DATA

The student should be able:

1. to organize ungrouped data in a frequency distribution table;

Example:

Twenty-five light bulbs were selected from a production lot of 1000 bulbs and tested for hours of bulb life. Construct a frequency distribution table to illustrate the test results.

803	896	878	886	861
880	976	912	955	843
871	843	895	875	882
883	927	942	918	843
939	890	868	881	901

"When confronted with a large amount of raw data, it is convenient to group the data into appropriate intervals. For easy computation, it is desirable to choose intervals with mid-points that are multiples of numbers such as five or ten. Usually there should be no fewer than ten intervals and no more than twenty-five intervals. In fixing the interval boundaries, it is customary to use one decimal place more than appears in the raw data so that each of the data will be contained in one and only one interval."[1]

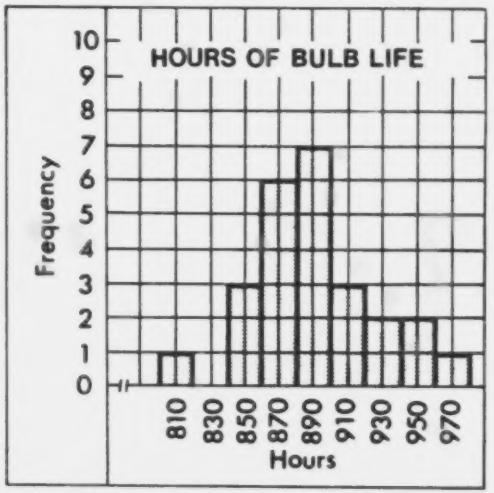
FREQUENCY DISTRIBUTION TABLE

Class Boundaries	Class Marks	Tally	Frequency	Class Mark times Frequency	Cumulative Frequency
959.5-979.5	970	I	1	970	25
939.5-959.5	950	II	2	1 900	24
919.5-939.5	930	II	2	1 860	22
899.5-919.5	910	III	3	2 730	20
879.5-899.5	890	IIII II	7	6 230	17
859.5-879.5	870	IIII I	6	5 220	10
839.5-859.5	850	III	3	2 550	4
819.5-839.5	830		0	0	1
799.5-819.5	810	I	1	810	1
			25	22 270	

[1] Willoughby, Stephen S., Probability and Statistics (Agincourt, ON, GLC Publishing Limited, 1972).

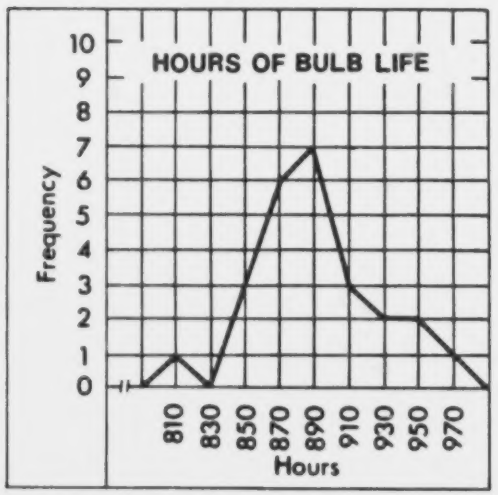
OBJECTIVES	NOTES TO TEACHERS

2. to construct a histogram to illustrate the information displayed in a frequency distribution table;



HISTOGRAM

3. to construct a frequency polygon to illustrate the information given in a frequency distribution table;



FREQUENCY POLYGON

OBJECTIVES	NOTES TO TEACHERS

2. to calculate from grouped data measure of central tendency including:

Calculate the mode, mean and median from the grouped data illustrated in the frequency distribution table.

FREQUENCY DISTRIBUTION TABLE

Class Boundaries	Class Mark x	Tally	Frequency f	Class Mark times Frequency	Cumulative Frequency
959.5-979.5	970	1	1	970	25
939.5-959.5	950	11	2	1 900	24
919.5-939.5	930	11	2	1 860	22
899.5-919.5	910	111	3	2 730	20
879.5-899.5	890	1111 11	7	6 230	17
859.5-879.5	870	1111 1	6	5 220	10
839.5-859.5	850	111	3	2 550	4
819.5-839.5	830		0	0	1
799.5-819.5	810	1	1	810	1
			25	22 270	

a) mode,

The mode is the most frequently recurring measure. In grouped data, measures lose their individuality and take on the characteristics of the class. The individual measures also become distributed evenly within a class. From the table, the mode is 890. In the raw data, the mode for the same measures was 843.

b) mean,

The mean $(\bar{x})$ for grouped data is obtained by multiplying the frequency of the class marks by the class mark and dividing the sum by the number of measures.

OBJECTIVES	NOTES TO TEACHERS

$$x = \frac{\sum(fx)}{N} = \frac{22\ 270}{25}$$ Where $\sum$ represents "the sum of"

$$\therefore = \underline{\underline{891}}$$

(The mean from the ungrouped data is 890.)

c) median;

The median of a set of grouped data is the measure of the middle element. In the set of 25 illustrated in the previous Frequency Distribution Table, the median is the measure of the 13th element. The cumulative frequency column shows that the 13th element is in the group whose class boundaries are 879.5 and 899.5 and would be the third of the 7 elements of $\frac{3}{7}$ of the width of the class (20).

$$Md = \frac{3}{7} \times 20 + 879.5 \therefore 8.57 + 879.5$$
$$\therefore 888.1$$

class width — lower limit

(The median from the ungrouped data is 883.)

Example:

Find the mean height and the median height of the 100 male students whose heights are summarized below.

Height	Class Mark	Frequency	Class Mark times Frequency	Cumulative Frequency
153.5–160.5	157	5	785	5
160.5–167.5	164	16	2 624	21
167.5–174.5	171	43	7 353	64
174.5–181.5	178	27	4 806	91
181.5–188.5	185	9	1 665	100
	TOTALS	100	17 233	

Solution:

To find mean:
1. calculate class mark
2. multiply class mark by frequency
3. total this column
4. divide this total by the total of the frequency

column $\frac{17\ 233}{100} = \underline{\underline{172.33}}$

To find median:
1. The median is between the 50th and 51st measures.
2. The sum of the first two class frequencies is 5 + 16 = 21, so to get the 50.5th measure we need 29.5 more measures from the next group.
3. The measure we want must be $\frac{29.5}{43}$ of the way through the interval.

OBJECTIVES	NOTES TO TEACHERS

4. Because the third class interval corresponds to the heights 167.5 to 174.5, the median is $\frac{29.5}{43}$ of the way between 167.5 and 174.5.

5. Therefore

$$Md = 167.5 + \frac{29.5}{43}(174.5 - 167.5)$$

$$\therefore \quad Md = 167.5 + 4.8$$

$$\therefore \quad Md = 172.3$$

5. to calculate from grouped data measures of dispersion (or variability) including:

a) range,

The range of the bulb life is the difference between the highest measure (970) and the lowest measure (810) which is 970 - 810 = 160 hours.

b) the standard deviation;

The standard deviation is the square root of the average of the squares of the differences of these measures from the mean.

Note: Standard deviation - usually denoted by σ (sigma) when dealing with a population and 's' when dealing with a sample.

$$\text{Standard deviation} = \sqrt{\frac{f(x-\bar{x})^2}{N}}$$

OBJECTIVES	NOTES TO TEACHERS

Example:

The heights of 100 children have been tallied with the resulting frequency distribution table shown below. Calculate the variance and standard deviation from this table.

Note: $\bar{x}$ = mean = 97.45

Class (cm)	Class Mark (x)	Frequency (f)	Deviation $x - \bar{x}$	$f(x-\bar{x})^2$
89.5- 92.5	91	5	-6.45	208.0125
92.5- 95.5	94	18	-3.45	214.2450
95.5- 98.5	97	42	-0.45	8.5050
98.5-101.5	100	27	2.55	175.5675
101.5-104.5	103	8	5.55	246.4200
		100		852.7500

Standard deviation = $\sqrt{\frac{852.750}{100}} = \sqrt{8.5275} \doteq 2.9$

6. to calculate from grouped data some measures of position including:

Measures of position are used to compare scores from different populations having different means and standard deviations.

a) the standard score or z-score,

The z-score is the number of standard deviations that a given score lies above or below the mean.

$$z = \frac{x - \bar{x}}{\sigma} \quad \text{or} \quad z = \frac{x - \mu}{\sigma}$$

where z = standard score
x or μ = mean
σ = standard deviation

Example:

If the mean score of a set of scores is 10 and the standard deviation of the same set of scores is 6.32, calculate:

a) the z-score of student A who had a raw score of 15.
b) the z-score of student B who had a raw score of 3.

Solution:

a) $z = \frac{x - \bar{x}}{\sigma}$

$= \frac{15 - 10}{6.32}$

$= 0.79$

This means tht student A was 0.79 standard deviations above the mean.

b) $z = \frac{x - \bar{x}}{\sigma}$

$= \frac{3 - 10}{6.32}$

$= - 1.11$

This means that student B was 1.11 standard deviations below the mean.

OBJECTIVES	NOTES TO TEACHERS
b) the percentiles for a given set.	The percentile is $\frac{1}{100}$ of a given set of elements arranged in order of magnitude. The 50th percentile corresponds to the median. Some test results (DAT, SAT, GRE, LSAT) are communicated in percentile only. Example 1: A student's score was ranked 75th on a test administered to 200 students. In which percentile would she be? Solution: If the student was ranked 75th then there would be 200 - 75 = 125 scores lower than hers. This would place her in the $\frac{125}{200}$ x 100% = 62.5 percentile. This means that approximately 62.5% of the students scored lower than she did. Example 2: If the students in Dick's class got the following scores on a test (54, 65, 71, 71, 71, 71, 74, 75, 76, 76, 77, 80, 80, 80, 92 and 98) and if the students in Jane's class got the following scores on the same test (41, 44, 56, 64, 66, 73, 74, 76, 76, 76, 78, 80, 80, 83, 84, 86, 88, 93, 95 and 98), what was the percentile rank of Dick and of Jane if each got a score of 76?

OBJECTIVES	NOTES TO TEACHERS

Percentile Rank of Score $x = \left(\frac{y + \frac{1}{2}z}{n}\right) 100$

Dick's percentile Rank $= \left(\frac{8 + \frac{1}{2}(2)}{16}\right)100 = 56.25$

Jane's Percentile Rank $= \left(\frac{7 + \frac{1}{2}(3)}{20}\right)100 = 42.5$

Note: Sometimes, results are given in quartiles or quintiles.

First quartile--approximately $\frac{1}{4}$ of the measures are less than or equal to it.

First quintile--approximately $\frac{1}{5}$ of the measures are less than or equal to it.

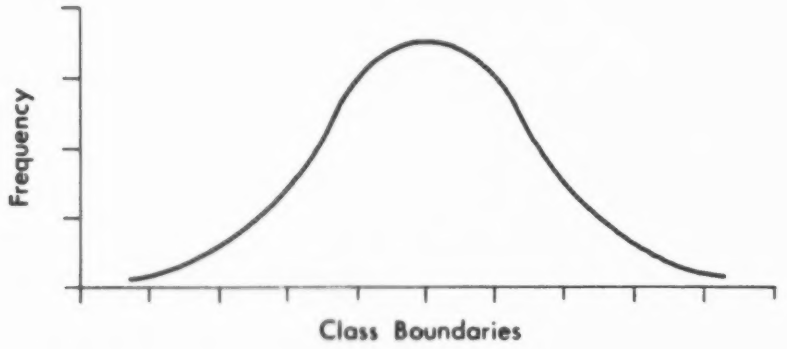
E. THE NORMAL DISTRIBUTION CURVE

The student will be able:

1. to examine some characteristics of the curve of normal distribution.

To introduce the normal distribution curve, ask students to sketch the graph of a frequency distribution with class boundaries denoted on the horizontal axis and frequency on the vertical axis. If the graph resembles a bell-shaped curve, the distribution is said to be "normal."

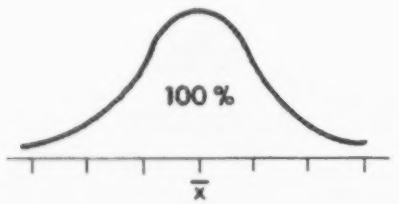
1. True False Test
2. Flip Coins
3. Roll Dice



Note: The frequency in any class interval can be represented by the area under the curve between the upper and lower limits of that interval.

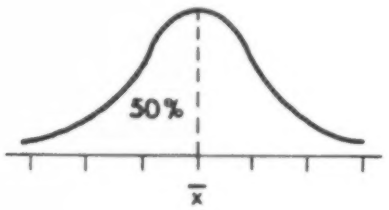
The following diagrams are useful in showing the relationship between a normal distribution and standard deviation (σ).

1. The total area under the normal curve is one and represents the total number of items to be considered.

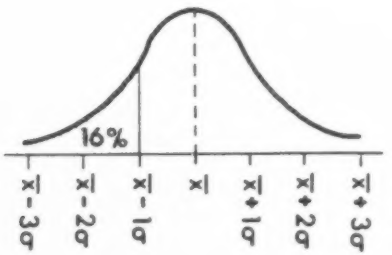


OBJECTIVES	NOTES TO TEACHERS

2. The mean ($\bar{x}$) corresponds to the 50th percentile, i.e., 50% of the items tested have test values less than the mean $\bar{x}$.

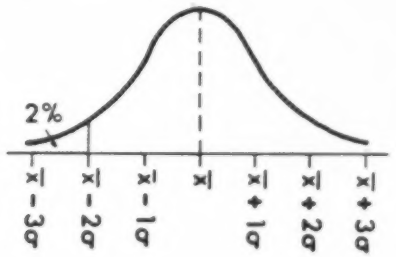


3. $\bar{x} - \sigma$ corresponds to the 16th percentile. 16% of the items tested have test values less than $\bar{x} - \sigma$. Conversely, 84% of the tested items have test values greater than $\bar{x} - \sigma$.

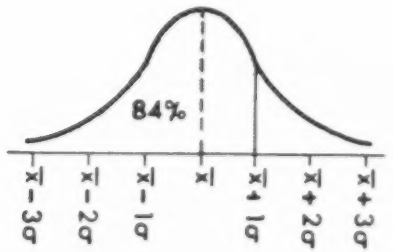


OBJECTIVES	NOTES TO TEACHERS

4. $\bar{x} - 2\sigma$ corresponds to the 2nd percentile, where 2% of the items tested have test values less than $\bar{x} - 2\sigma$. Conversely, 98% of the tested items have test values greater than $\bar{x} - 2\sigma$.



5. $\bar{x} + \sigma$ corresponds to the 84th percentile, where 84% of the items tested have test values less than $\bar{x} + \sigma$. Conversely, 16% of the tested items have test values greater than $\bar{x} + \sigma$.

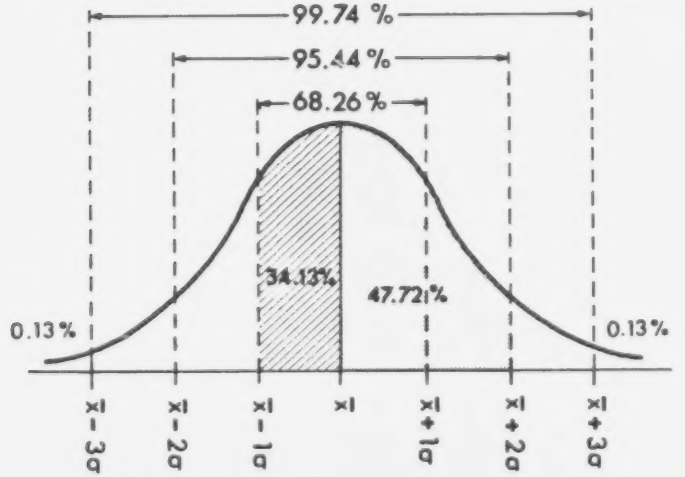


81

82

The following table summarizes the relationship for various test values in terms of $\bar{x}$ and σ:

Test Values	Percentage of Items Less Than Test Values
$\bar{x} - 3\sigma$	.13%
$\bar{x} - 2\sigma$	2
$\bar{x} - 1.5\sigma$	7
$\bar{x} - 1\sigma$	16
$\bar{x} - 0.5\sigma$	30
$\bar{x}$	50
$\bar{x} + 0.5\sigma$	70
$\bar{x} + 1\sigma$	84
$\bar{x} + 1.5\sigma$	93
$\bar{x} + 2\sigma$	98
$\bar{x} + 3\sigma$	99.87%



$\bar{x}$ - represents the mean

σ represents standard deviation

In the above sketch, 68.26% of the measures lie within one standard deviation of the mean. This would mean, for example, that if a set of marks has a mean of 80 and a standard deviation of 9, then 68.26% of the marks are between 71 and 89 (80±9).

Examples:

1. The Sparky Electric Company advertises bulbs to burn at least 1000 hours. A sample of bulbs has a mean life of 1050 hours and a standard deviation (σ) of 25 hours.

   a) What percent of bulbs will last at least 1000 hours?
   b) What percent of bulbs will last more than 1075 hours?

   Solution:

   a) $x = 1050 \qquad \sigma = 25$
   1000 hours is $1050 - 50 = x - 2\sigma$
   but $x - 2$ (see table) represents the 2nd percentile
   ∴ 2% of the bulbs will last less than 1000 hours
   ∴ 98% of the bulbs will last at least 1000 hours

   b) 1075 hours is $1050 + 25 = x + 1\sigma$, which represents the 84th percentile
   ∴ 84% will last less than 1075 hours
   ∴ 16% will last more than 1075 hours

2. A student scored 690 on an aptitude test. The mean of the test is 600 with a standard deviation of 60. What percent of the students scored lower than 690?

Solution:

$\bar{x} = 600$, $\sigma = 60$
690 is $600 + 90 = x + 1.5\sigma$
$x + 1.5\sigma$ corresponds to the 93rd percentile
93% of the students scored lower than this student
7% of the students scored higher than this student

General Questions by Objectives for the Following Data Sets

Objective: Organize each data set into a frequency distribution table. (As assigned.)

Objective: Using the frequency distribution table, prepare histograms and frequency polygons for each data set. (Students could use different interval sizes and compare results.) (As assigned.)

Objective: Calculate the mode, mean, median, and mean derivation for each data set. (As assigned.)

Objective: Calculate the range, the mean deviation, the variance and the standard deviation for each data set. (As assigned.)

Objective: Calculate percentiles for one of the data sets. Construct a cumulative frequency graph and read the quartiles and quantiles from it.

Objective: Use your knowledge of the characteristics of the normal curve to answer questions of the type shown on the preceding page.

Data Sets (Data supplied)

The accompanying data sets have been constructed so there are sets with the same mean and different dispersions and sets with different means and the same dispersions. Possible interpretations are given as constructed experiments to create more realistic word problems. The same data set and problem(s) can be referred to for most of the objectives.

1. Weight of 100 professional football players measured in kg.
   Mean $\doteq$ 100.7, $\sigma \doteq$ 7.7

110	103	88	102	85	116	98	92	105	109
97	105	111	84	93	107	101	92	109	88
91	103	104	99	105	96	96	106	111	104
97	108	101	95	112	96	104	107	87	101
98	110	98	107	101	101	113	108	101	91
116	96	105	105	105	95	99	90	110	94
93	94	100	107	114	112	95	91	100	101
97	101	99	98	104	96	98	81	105	103
104	90	110	106	96	80	103	106	97	93
105	102	111	108	93	102	96	116	103	104

2. Each of 100 workers are told to cut a piece of wood 100 mm long. The actual measurements are identified below.
   Mean $\doteq$ 100.1; $\sigma \doteq$ 0.8

101	100	98.8	100.2	98.5	101.6	99.8	99.2	100.5	100.9
99.7	100.5	101.1	98.4	99.3	100.7	100.1	99.2	100.9	98.8
99.1	100.3	100.4	99.9	100.5	99.6	99.6	100.6	101.1	100.4
99.7	100.8	100.1	99.5	101.2	99.6	100.4	100.7	98.7	100.1
99.8	100	99.8	100.7	100.1	100.1	101.3	100.8	100.1	99.1
101.6	99.6	100.5	100.5	99.9	99.5	99.9	99	101	99.4
99.3	99.4	100	100.7	101.4	101.2	99.5	99.1	100	100.1
99.7	100.1	99.9	99.8	100.4	99.6	98.1	98.1	100.5	100.3
100.4	99	100.6	100.6	99.6	98	100.6	100.6	99.7	99.3
100.5	100.2	100.8	100.8	99.3	100.2	101.6	101.6	100.3	100.4

3. 100 people buying coffee put the following amounts of sugar (grams) in their coffee.
Mean ≐ 10.1, σ ≐ 0.8

11	10.3	8.8	10.2	8.5	11.6	9.8	9.2	10.5	10.9
9.7	10.5	11.1	8.4	9.3	10.7	10.1	9.2	10.9	8.8
9.1	10.3	10.4	9.9	10.5	9.6	9.6	10.6	11.1	10.4
9.7	10.8	10.1	9.5	11.2	9.6	10.4	10.7	8.7	10.1
9.8	11	9.8	10.7	10.1	10.1	11.3	10.8	10.1	9.1
11.6	9.6	10.5	10.5	9.9	9.5	9.9	9	11	9.4
9.3	9.4	10	10.7	11.4	11.2	9.5	9.1	10	10.1
9.7	10.1	9.9	9.8	10.4	9.6	9.8	8.1	10.5	10.3
10.4	9	11	10.6	9.6	8	10.3	10.6	9.7	9.3
10.5	10.2	11	10.8	9.3	10.2	9.6	11.6	10.3	10.4

4. Weights (kg) of a group of 13 year-olds in a judo competition.
Mean ≐ 49.4, σ ≐ 4.4

53.5	45	54	45.5	51.5	58	49	53	42	53.5
49.5	44.5	45.5	52.5	52	48.5	44	51	54.5	59.5
46	40	51.5	51.5	43.5	49.5	48	44.5	54.5	58.5
45.5	46.5	57.5	51.5	49.5	47	48.5	49	47.5	49
52.5	51.5	51	47	49.5	49	49.5	46	51	45
45.5	52	48	49.5	50.5	48	51	48.5	51.5	47
50.5	47	42.5	43	44	51.5	44.5	46.5	48	48.5
45.5	44.5	54.5	50.5	50.5	57	46.5	50	47.5	52.5
57.5	55	49.5	56.5	56.5	46	48	53	51	45
52	49	46.5	40	40	41.5	47	55.5	56	52

5. Length of time (days) that certain items sit on a store shelf. (Could be used to describe the rate of turnover for merchandise.)
Mean ≐ 38.9, σ ≐ 4.1

38	42.4	37.5	41.2	41.6	38.8	42	43.2	44.8	42.8
37.6	38.4	46.4	38	44	39.2	30.8	38	48	37.6
36.4	37.6	41.2	35.6	38.4	40.8	46	47.2	42	42.4
36.8	39.2	45.2	48	37.6	44	40.4	40.8	41.2	39.2
46.4	34	43.2	39.6	36.4	38	39.6	46.4	35.6	47.6
38.4	45.6	32	42.4	38.8	38	32	40.4	38.8	34.8
35.6	39.2	38	33.6	38.4	38.4	41.6	42	37.2	40.8
35.6	36	38.4	38.4	39.2	46.4	37.2	46	40.4	44
42.8	40.8	37.6	41.6	38.8	38.8	44	43.2	43.6	40.8
39.6	41.2	38	36	42	45.6	34.4	32	38	36

6. Cost in cents of 100 meals at a fast-food outlet.
Mean ≐ 495, σ ≐ 48

475	530	470	515	520	485	525	540	560	535
470	480	580	475	550	490	485	475	600	470
455	470	515	445	480	510	575	590	525	530
460	490	565	600	470	550	505	510	515	490
580	425	540	495	455	475	495	580	445	595
480	570	400	530	485	475	400	505	485	435
445	490	475	420	480	480	520	525	465	510
445	450	480	480	490	580	465	575	505	550
535	510	470	520	485	485	550	540	545	510
495	515	475	450	570	570	430	400	475	450

OBJECTIVES	NOTES TO TEACHERS

Data Sets (Data to be generated by student.)

1. Measure the length of a desk.
2. Identify what is deemed to be a 10-second period.
3. Guess the number of beans in a jar.
4. Measure heights of students.
5. Count the number of paces to cover a block in front of the school.
6. Count the number of cars passing the school during specific periods of time.
7. Record the total of grocery bills at several checkouts in a supermarket.

Note: Generate a minimum of 30 pieces of data.

TOPIC III: NUMBER THEORY (1 MODULE)

OBJECTIVES	NOTES TO TEACHERS

The student should be able:

1. to determine the validity of a mathematical formula by the principle of Mathematical induction:

   Prove the formula is valid for n = 1.

   Assuming formula to be valid for n = k, show formula remains valid for n = k+1. Since k is arbitrary, the formula must be valid for all n;

Notation:

$I- = \{\ldots, -3, -2, -1\}$ (negative integers)
$I+ = \{1, 2, 3, \ldots\}$ (positive integers)
$W = \{0, 1, 2, \ldots\}$ (whole numbers)
$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (integers)

Examples:

a) $1 + 2 + 3 + \ldots + n = \frac{n\,(n + 1)}{2}$

b) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

c) $1 + 8 + 27 + \ldots + n^3 = \frac{n^2\,(n + 1)^2}{4}$

d) $2(3^0) + 2(3^1) + 2(3^2) + \ldots + 2(3^n - 1$

e) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Notes on Induction:

a) Illustrate the $1 + 2 + \ldots + n = \frac{n(n + 1)}{2}$

$$\begin{array}{r} 1 + 2 + 3 + 4 + 5 \\ \underline{5 + 4 + 3 + 2 + 1} \\ 6 + 6 + 6 + 6 + 6 \end{array} \qquad \ldots \text{ Gauss' Method}$$

So, twice the sum = $30 = 5(5 + 1) = 5(6)$

$$\text{sum} = 15 = \frac{5(5 + 1)}{2} = \frac{5(6)}{2}$$

In general, $1 + 2 + \ldots + n = \frac{n(n + 1)}{2}$

The Moessner process or the Domino principle are examples of the use of Mathematical Induction. (See page 56, resource #2.)

Induction Proof:

Step 1: Show the formula is valid for n = 1.

$$\frac{n(n + 1)}{2} = \frac{1(2)}{2} = 1$$

$\therefore$ Formula is valued for n = 1.

Step 2: Assume formula is valid for n = k.

prove the formula remains valid when n = k + 1.

Assumption: $1 + 2 + \ldots + k = \frac{k(k + 1)}{2}$

Form the expression: $1 + 2 + \ldots + k + k + 1$.
From the assumption this new expression is

$$\frac{k(k + 1)}{2} + (k + 1)$$

Factoring: $(k + 1)\left(\frac{k}{2} + 1\right)$

$$(k + 1)\left(\frac{k + 2}{2}\right)$$

This agrees with the formula. Since k is arbitrary, the formula must be true for all values of n.

OBJECTIVES	NOTES TO TEACHERS
	b) Moessner used a process of crossing out every second integer: 1 2 3 4 5 6 7 8 9 10 11 12 13 . . . to obtain: 1 3 5 7 9 11 12 13 15 17 19 . . .. Keep adding the integers and forming partial sums: 1 4 9 16 15 36 49 64 81 100 . . .. We conjecture $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ Formal induction can be used to prove the conjecture. Caution students that some formulae are valid only on a limited basis. For example, the formula $n^2 - n + 41$ generates a prime number for n = 1 to 40 but FAILS when n = 41. This counter example shows the necessity for formal proof.
2. to state the Archimedian property;	For all a, b, $a > b \in I^+$, there exists an $n \in I^+$ such that $bn > a$ Example 1: a = 21, b = 5 if n = 5, bn = 5(5) = 25 > 21 = a

OBJECTIVES	NOTES TO TEACHERS
	Example 2:
	Given a = 27 and b = 5, find $n \in I^+$ so that bn > a. If n = 6, then bn > a, i.e., 6(5) > 27.
	This is a use of the Archimedian Principle.
3. to illustrate the division algorithm:	
$a \in W$, $b \in I^+$ there exists $q \in W$ such that $a = bq + r$, $0 < r < b$;	Example:
	a = 42, b = 5
	42 = 5(8) + 2 So q = 8 and r = 2 < 5 Note: Could also use $\frac{42}{5}$ = 8 Remainder 2
4. to determine if d is (is not) a divisor of m:	
d | m if there is a q for which m = qd,	d | m: d is a divisor of m d | m: d is not a divisor of m
d | m if there is no q for which m = qd;	Example:
	Show if a | b and a | c and b > c, then a | (b-c)
	Show students a specific example:
	Let a = 2, b = 18, c = 12
	2 | 18 and 2 | 12 then 2 | 6 2 | 18 implies 18 = 2(9) 2 | 12 implies that 12=2(6) 6=18-12=2(9)-2(6)=2(9-6) =2(3)

OBJECTIVES	NOTES TO TEACHERS
	Hence 2|6 A similar proof can be written for the general case.
5. to define a common divisor of a and b;	d is a common divisor of a and b if d|a and d|b.
6. to define the Greatest Common Divisor (GCD) of a and b denoted by GCD (a,b);	g is the GCD of a and b if: (a) g is a common divisor of a,b (b) g is divisible by EVERY common divisor of a,b
7. to find the GCD of 2 integers for a simple case;	Example: GCD (15, 20) = 5
8. to express GCD (a,b) in the form GCD (a,b)=ka + mb where k, m ∈ I, for a simple case;	Example: GCD (15, 20) = -1(15) + 1(20) = 5 ↑ ↑ ↑ ↑ a b k m
9. to find GCD (a,b);	Example: Find GCD (54, 28). 54 = 28(1) + 26 28 = 26(1) + 2 26 = 2(13) + 0, implying that 2 = GCD (54, 28). This process is called the Euclidean Algorithm.

OBJECTIVES	NOTES TO TEACHERS

10. to express GCD (a,b) in the form GCD (a,b) = ka + mb where k, m ∈ I, for more difficult cases;

Example:

$$2 = 28 - 26 \text{ But}$$
$$26 = 54 - 28$$
$$2 = 28 - (54 - 28)$$
$$= 56 - 54$$
$$= 2(28) - 54 = \underset{\uparrow\ \downarrow}{-1}(54) + \underset{\uparrow\ \downarrow}{2}(28)$$
$$\qquad\qquad\qquad\quad k\ a \qquad m\ b$$
$$= k(a) + m(b)$$

ENRICHMENT

If p is a prime number and p|a, b and p|a, then p|a.

11. to define "relatively prime" integers;

a is relatively prime to b if GCD (a,b) = 1

12. to define "prime number";

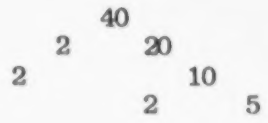
A number p is "prime" if there is no k where $2<k<p-1$ for which k|p

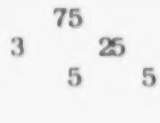
Example:

Let M = 100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

OBJECTIVES	NOTES TO TEACHERS
	Circle 2 and cross out all multiples of 2 Circle 3 and cross out all multiples of 3 Circle 5 and cross out all multiples of 5 Circle 7 and cross out all multiples of 7 Numbers left are primes. For 1 to M, cross out <u>all</u> multiples of primes $\leq \sqrt{M}$.
14. to prove there are infinitely many primes;	Assume there are only n primes $P_1, P_2 \cdot \cdot \cdot P_n$ The assumption is contradicted by the "new" number $(P_1)(P_2) \cdot \cdot \cdot (P_n) + 1$
15. to state the Fundamental Theorem of Arithmetic (FTA), i.e., every non-zero integer has a unique factorization into primes;	Teachers may wish to prove the Fundamental Theorem of Arithmetic. Note that if an integer is NOT prime, it must be divisible by a prime. To demonstrate unique factorization, assume distinct factorizations and show both are identical.
16. to define Lowest or Least Common Multiple (LCM) (a,b);	If a and b are uniquely prime factored then: LCM (a,b) = product of highest power for each different prime factor.
17. to use the "FTA" to find the LCM (a,b).	<u>Example</u>: Find LCM (40, 75)





$40 = 2^3 \times 5$ (unique)

$75 = 3 \times 5^2$ (unique)

LCM $(40,75) = 2^3 \times 3 \times 5^2 = 600$

(Note: different factors: 2, 3, 5)

ENRICHMENT

Teachers may wish to make students aware of the following unproven conjectures.

1. Goldbach's Conjecture (unproven): Every even number > 4 is the sum of 2 prime numbers greater than 2.

2. Fermat's Last Theorem (unproven): There are no solutions (integer) to $x^n+y^n=z^n$ where $n > 2$ and integral.

Students may use a computer to test the conjectures.

Resources

1. Seymour, I, and Shedd, M. Finite Differences: Creative Publications, 567 Clarke Road, Coquitlam, BC V3J 3X4, 1981.

2. For information on the Moessner Process (p. 44) write to: Professor Calvin T. Long, Mathematics Department, Washington State University, Pullman, Washington.

TOPIC IV: LINEAR PROGRAMMING ($\frac{1}{2}$ MODULE)

OBJECTIVES	NOTES TO TEACHERS

The student should be able:

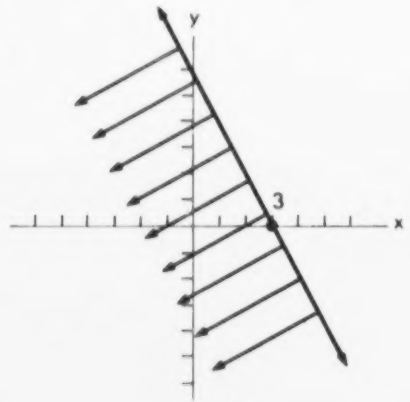
1. to graph open or closed half-planes;

Examples:

a) $2x + 6 \leq 6$;

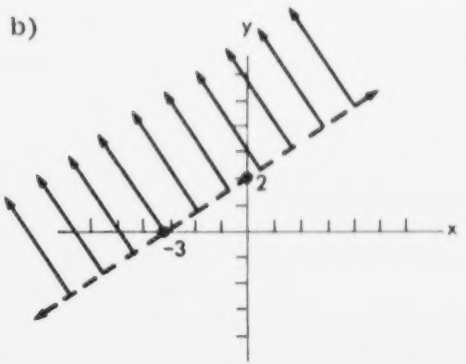
b) $2x - 3y < -6$

a)

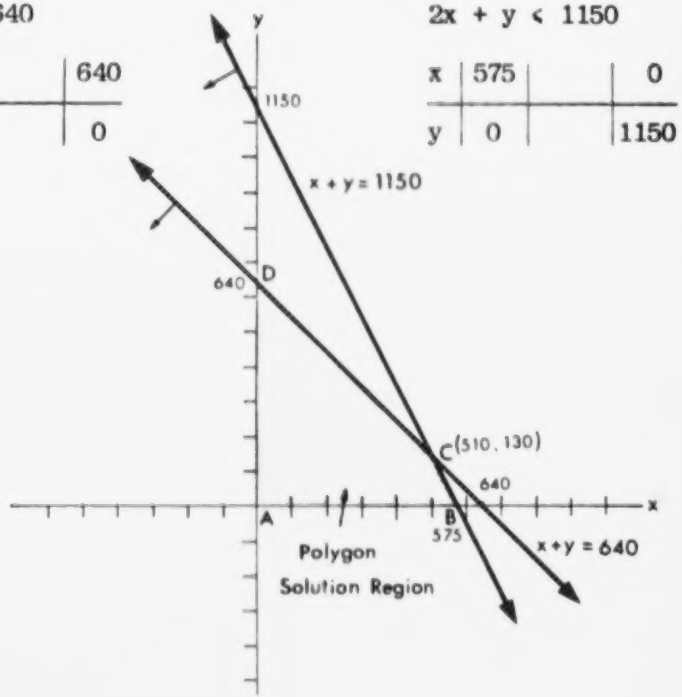


$2x + y \leq 6$
("closed" half-plane)

b)



$2x - 3y < -6$
("open" half-plane)

OBJECTIVES	NOTES TO TEACHERS
2. to graph a system of inequalities and shade the solution region;	Example: $x + y \leq 640$ $2x + y \leq 1150$ $x \geq 0,\ y \geq 0$ Since $x \geq 0$ and $y \geq 0$, solution is in quadrant 1 only. $x + y \leq 640$ 	x	0		640
---	---	---	---
y	640		0 | $2x + y \leq 1150$ | x	575		0
---	---	---	---
y	0		1150	  A (0,0) C (510,130) B (575,0) D (0,640)

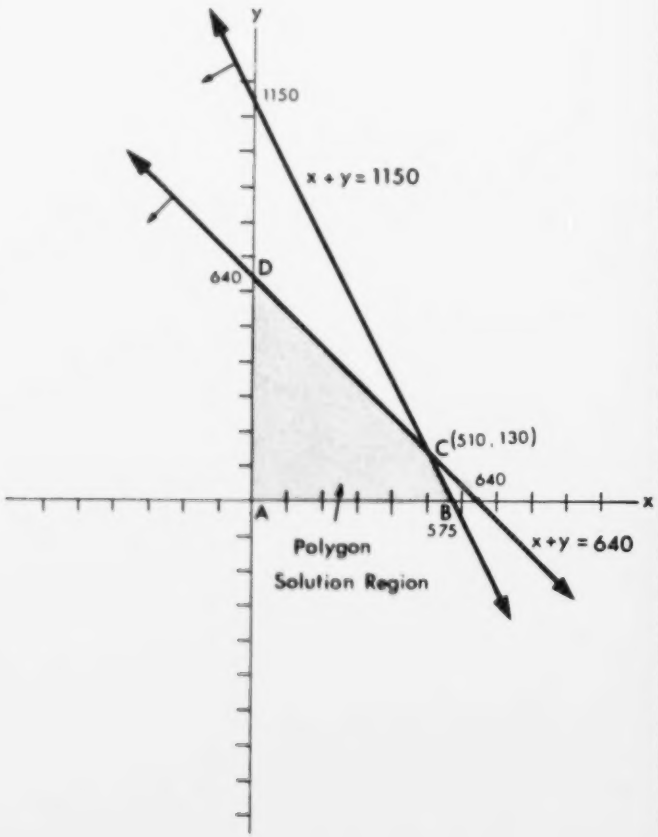
OBJECTIVES	NOTES TO TEACHERS

3. to determine the solution set algebraically;

$x + y \leq 640$
$2x + y \leq 1150$

Solve the equalities

$x + y = 640$
$x + y = 1150$
$x = 510$
$y = 130$
$C(510,130)$



A (0,0) C (510,130)
B (575,0) D (0, 640)

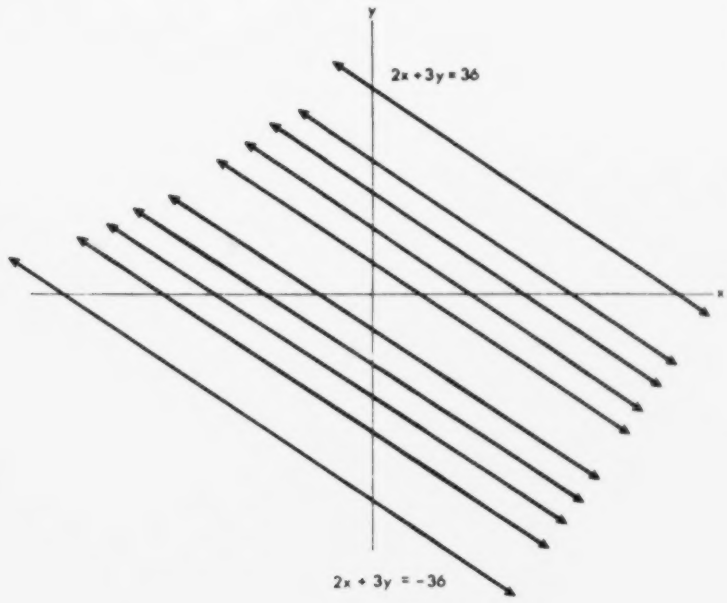
Maximizing, Minimizing a Function Over a Given Region

This following example will show why the maximum or minimum value of a given linear function over a bounded polygonal region occurs at one of the vertices.

Example:

Graph (some of) the family of lines determined by:

$2x + 3y = k$ where $k \in I$



A FAMILY of LINES, all having slope - 2/3

Values of the Function: $2x+3y=k$

k	x intercept	y intercept
-36	-18	-12
-24	-12	-8
-18	-9	-6
-12	-6	-4
-6	-3	-2
6	3	2
12	6	4
18	9	6
24	12	8
36	18	12

Maximum Value: Locate the point of the region where the graph of a member of the "Family" produces the highest y intercept of function lines.

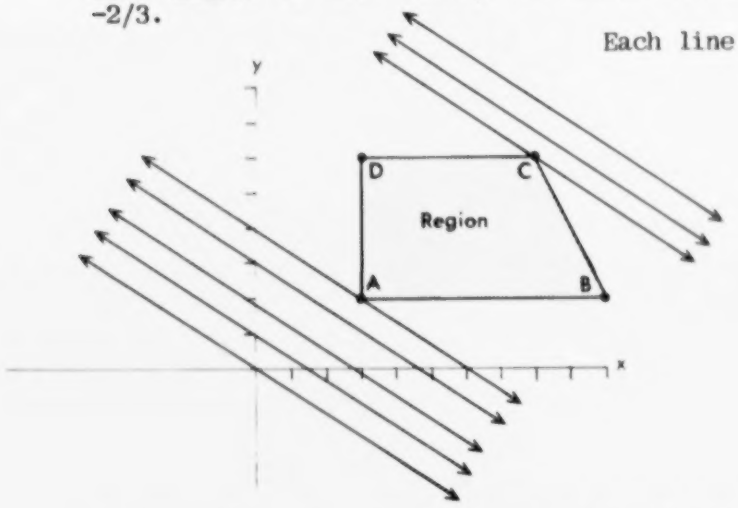
Minimum Value: Locate the point of the region where the graph of a member of the "Family" of function lines produces the lowest y intercept.

Example:

Maximize, Minimize $2x + 3y = k$ over a defined region, ABCD.
A = (3,2) B = (10,2) C = (8,6) D = (3,6)

Solution:

Each member of the "Family" of lines defined by $2x + 3y = k$ has slope = $-2/3$.



VERTEX	K = 2x + 3y
A=(3,2)	12 (Minimum)
B=(10,2)	26
C=(8,6)	34 (Maximum)
D=(3,6)	24

The line $2x + 3y = 12$ produces "minimum" contact point.
The line $2x + 3y = 34$ produces "maximum" contact point.

OBJECTIVES	NOTES TO TEACHERS

John Von Neumann proved that the minimum or maximum of a linear function occurs at one of the vertices of the polygonal solution region.

4. to solve a linear programming problem, by maximizing or minimizing a given linear function over a region defined by a set of constraints.

Maximize or minimize a given linear function over a region defined by a set of constraints. See a graph opposite Objective #2 of this topic.

Example:

Maximize $k = 20x + 35y$ and minimize $L = 10x - 3y$ over the region defined by:

$$x + y \leq 640$$
$$2x + y \leq 1150 \quad \text{constraints}$$
$$x \geq 0,\ y \geq 0$$

Vertex	K=20x + 35y	L=10x - 3y
A=(0,0)	0	0
B=(575,0)	11 500	5750
C=(510,130)	10 200 + 4550 = 14 750	4710
D=(0,640)	22 400	-1920

K is maximized by the point D = (0,640) (minimized by A)
L is minimized by the point D = (0,640) (maximized by B)

5. to translate a verbal problen into a linear programming problem and solve;

Example:

An oil refinery has a maximum production of 5000 barrels/day. It produces 2 types of oil: type "G" for gasoline, type "D" for diesel fuel.

Constraints: at least 750 barrels of "D" must be produced each day.

If profit is \$750/barrel for "G" and \$5.00/barrel for "D," find the maximum profit/day.

Solution:

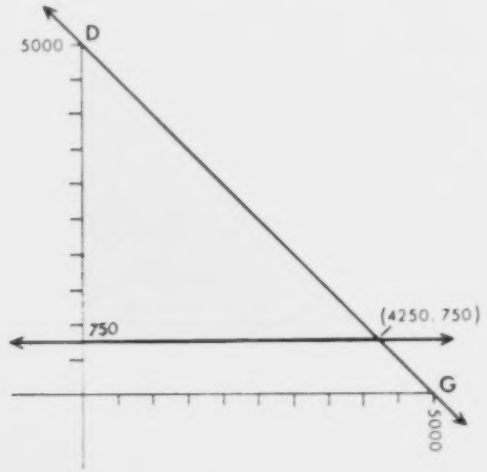
Profit Function:

P = 7.50 G + 5.00 D where
G = # barrels of type "G" (per day)
D = # barrels of type "D" (per day)

Constraints:

G ≥ 0, D ≥ 0
G + D ≤ 5000, D ≥ 750

VERTEX	P VALUE
(0,750)	3 750
(4250,750)	35 625
(0,5000)	25 000



Maximum profit occurs at (4250,750)
Here, P = 31 875 + 3 750 = $35 625 per day, if the refinery produces 4250 barrels of type "G" and 750 barrels of type "D."

Example 2:

To produce chrome-vanadium steel, Titanium Corporation uses a combination of chrome and vanadium. Cost per unit and weight per unit of chrome-vanadium steel must all be considered for different industrial applications. Information about the use of the steel in car doors is summarized below:

Metal	Cost/100 g of steel ($)	Weight/100 g of steel
1 cm^2 chrome	5.00	12 g
1 cm^2 vanadium	9.00	18 g

The corporation is limited to a maximum cost of $250 per 100 g of steel. Also, no more than 8 cm^2 of chrome nor less than 20 cm^2 of vanadium must be used (government requirements). How many square cm of each metal must be used to maximize weight?

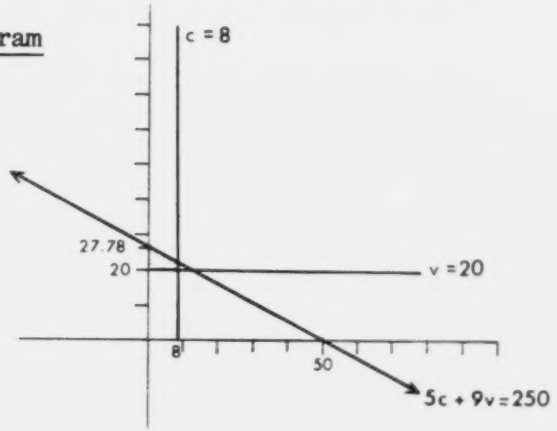
Solution:

c = # cm^2 of chrome used
v = # cm^2 of vanadium used

Objective Function: $W = 12c + 18v$ (to be maximized)

Constraints: $0 \leq c \leq 8$
$v \geq 20$
$5c + 9v \leq 250$

Diagram



Vertex	W = 12c + 18v
(0,20)	360
(0,27.78)	500.04
(8,20)	456
(8,23.33)	515.94

At 8 cm^2 of chrome and 23.33 cm^2 of vanadium, we have maximum weight per 100 kg of steel.

Summary of Linear Programming Techniques

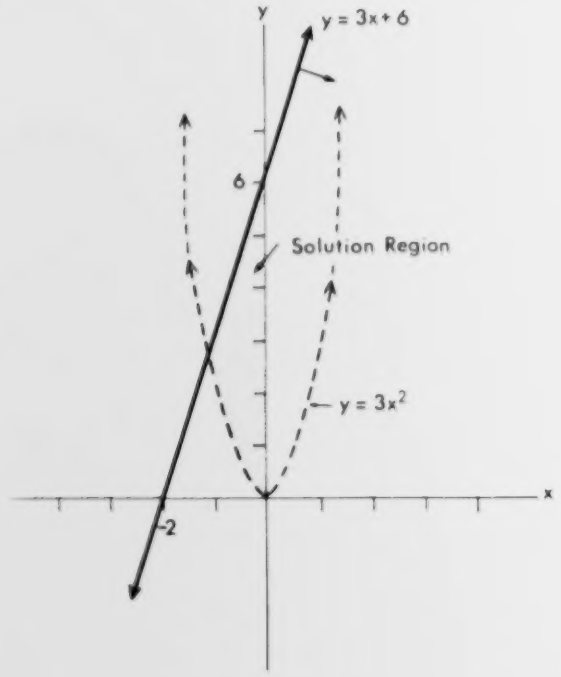
1. Define variables.
2. Write a system of inequalities.
3. Graph the system. Find vertices of the polygon formed.
4. Write an expression to be maximized or minimized.
5. Substitute values from vertices into the expression.
6. Select the greatest or least result.

ENRICHMENT

1. Solve a non-linear of inequalities:

Example: $y \geq 3x^2$

$y \leq 3x + 6$

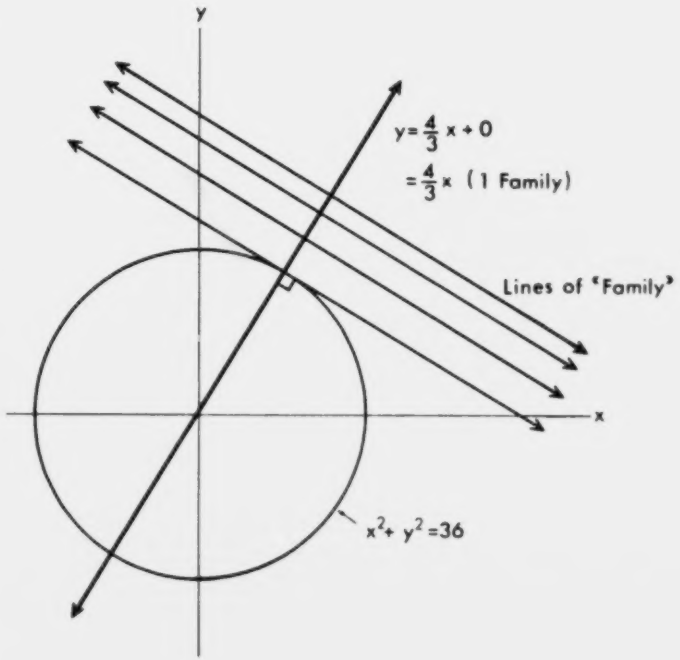


1:7

1:8

2. Maximize $3x + 4y$ over the region defined by $x^2 + y^2 \leq 36$.

Solution:



Each line in "Family" has slope = $-3/4$

$y = \frac{4}{3}x + 0$

$= \frac{4}{3}x$ (1 Family)

The Point Needed:

Where $y = \frac{4}{3}x$ and $x^2 + y^2 = 36$ cross

Solving: $x^2 + \frac{9}{16}x^2 = 36$

$\frac{25}{16}x^2 = 36$

$x^2 = 23.04$
$x = 4.8$
$y = 6.4$

The point = (4.8,6.4). Substituted in $3x + 4y$ yields 40 for k.

Note the point = (-4.8, -6.4) would yield the minimum of $3x + 4y$ over the region.

TOPIC V: THEORY OF EQUATIONS (2 MODULES)

The primary objective of this topic is to learn how to solve polynomial equations. Students are acquainted with the methods of solving linear and quadratic equations, but how would they go about solving $x^7 - 6x^4 + 3x - 1 = 0$?

It is interesting to see how students react when asked to solve such an equation. Apart from the typical "we haven't learned that yet" answer, some students will comment on formulae methods and some may consider factoring. Evariste Galois (1811-1832) proved that for polynomial equations of degree greater than four, there was no single method for obtaining all the solutions.

It would be reasonable to have students consider an equation of the type $(x + 2)(x - 3)(x + 5)(x - 6)(x - 1) = 0$. The concept of zeroes as geometric intercepts of the x-axis and roots as algebraic solutions for the equation can be linked using this equation as an example.

OBJECTIVES	NOTES TO TEACHERS
The student should be able:	
1. to add, subtract, multiply and divide polynomial expressions;	See Mathematics 100, C.II. Review as necessary.
2. to factor certain polynomial expressions;	See Mathematics 100, C.III, Mathematics 200, C.IV and Mathematics 300, CI. Review as necessary.
3. to solve quadratic equations;	See Mathematics 200, C.V. Review as necessary.
4. to divide a polynomial P(x) by a linear polynomial L(x) using the method of synthetic division;	Notes: 1. This kind of division is best developed and explained by use of an example. 2. Only linear polynomial divisors of the form $x \pm r$ should be considered at this time.

OBJECTIVES	NOTES TO TEACHERS

Example:

Divide $2x^4 - 5x^3 + 2x - 1$ by $x - 4$

a) by ordinary long division
b) by abridged long division (writing coefficients only)
c) by abridged method omitting unnecessary terms
d) by synthetic division

a) By Ordinary Long Division

$$\begin{array}{rl} & 2x^3 + 3x^2 + 12x + 50 \text{ (Quotient)} \\ \text{(Divisor) } x - 4 \;\big| & 2x^4 - 5x^3 + 0x^2 + 2x - 1 \text{ (Dividend)} \\ & \underline{2x^4 - 8x^3} \\ & \quad 3x^3 + 0x^2 \\ & \quad \underline{3x^3 - 12x^2} \\ & \qquad 12x^2 + 2x \\ & \qquad \underline{12x^2 - 48x} \\ & \qquad\quad 50x - 1 \\ & \qquad\quad \underline{50x - 200} \\ & \qquad\quad (+199) \text{ Remainder} \end{array}$$

OBJECTIVES	NOTES TO TEACHERS

b) By Abridged Long Division (Writing Coefficients Only)

$$
\begin{array}{r|l}
 & 2 + 3 + 12 + 50 \\
\hline
1 - 4 & 2 - 5 + 0 + 2 - 1 \\
 & \underline{2 - 8} \\
 & \quad 3 + 0 \\
 & \quad \underline{3 - 12} \\
 & \quad\quad 12 + 2 \\
 & \quad\quad \underline{12 - 48} \\
 & \quad\quad\quad 50 - 1 \\
 & \quad\quad\quad \underline{50 - 200} \\
 & \quad\quad\quad\quad (+199)
\end{array}
$$

c) By Abridged Method Omitting Unnecessary Terms

$$
\begin{array}{r|l}
1 - 4 & 2 - 5 + 0 + 2 - 1 \\
\hline
 & \quad - 8 \\
 & \quad\quad 3 \\
 & \quad\quad - 12 \\
 & \quad\quad\quad 12 \\
 & \quad\quad\quad - 48 \\
 & \quad\quad\quad\quad 50 \\
 & \quad\quad\quad\quad - 200 \\
 & \quad\quad\quad\quad (+199)
\end{array}
$$

Quotient: $2x^3 + 3x^2 + 12x + 50$; Remainder: $(+199)$

d) By Synthetic Division

Bring all the work into three lines.
Omit the first term of the divisor.

$$\begin{array}{r|rrrrr} -4 & 2 & -5 & +0 & +2 & -1 \\ & & -8 & -12 & -48 & -200 \\ \hline & 2 & +3 & +12 & +50 & (+199) \end{array}$$

Quotient: $2x^3 + 3x^2 + 12x + 50$; Remainder: $(+199)$

The numbers of the third line are determined by subtracting the numbers of the second line from those of the first line. The third line gives the coefficients of the quotient and the remainder in parentheses.

Key Steps in the Synthetic Division Process

1. Arrange the coefficients of P(x) in order of descending powers of x. (Write 0 as the coefficient of each missing power.)

2. After writing the divisor in the form (x - r), use "r" to generate the second and third rows of numbers as follows:

   Bring down the first coefficient of the dividend and multiply it by "r", then add the product to the second coefficient of the dividend. Multiply this sum by "r" and add the product to the third coefficient in the dividend. Repeat the process until a product is added to the constant term of P(x).

   Note: This process is well-suited to calculator use. Store "r" then proceed from left to right, recalling "r" and using it as indicated.

OBJECTIVES	NOTES TO TEACHERS

3. The last number in the third row of numbers is the remainder; the other numbers in the third row are the coefficients of the quotient, which is of degree "1" less than P(x).

   Example:

   Divide $(3x^4 - 4x^3 + x^2 - 12x + 3)$ by $(x - 2)$

   Using synthetic division

$$\begin{array}{r|l} -2 & 3 - 4 + 1 - 12 + 3 \\ & \quad -6 - 4 - 10 + 4 \\ \hline & 3 + 2 + 5 - 2 + (-1) \end{array}$$

   Quotient: $(3x^3 + 2x^2 + 5x - 2)$; Remainder: (-1)

5. to write the result of long division $\frac{P(x)}{L(x)}$ in the form $\frac{P(x)}{L(x)} = Q(x) + \frac{R}{L(x)}$ where R is a constant.

   This result is called Division Algorithm, which frequently is written in the form $P(x) = Q(x)\,L(x) + R$, where $Q(x)$ is called the quotient, $L(x)$ the divisor and R the remainder.

   Example:

$$\frac{4x^5 - 30x^3 - 50x - 6}{x + 3} = 4x^4 - 12x^3 + 6x^2 - 18x + 4 + \frac{-18}{x + 3}$$

6. to prove the Remainder Theorem;

   The Remainder Theorem states: if R is the remainder after dividing the polynomial P(x) by $x - r$, then $P(r) = R$.

OBJECTIVES	NOTES TO TEACHERS

Proof: From the Division Algorithm

$$P(x) = (x - r)\,Q(x) + R$$
If we let $x = r$, then
$$P(r) = (r - r)\,Q(r) + R$$
$$= 0 + R$$
$$= R$$

Its usefulness can be demonstrated quickly to students by considering the following:

Suppose we want to find the remainder R when $P(x) = x^{35} + x^{17} - 5$ is divided by $x + 1$. We could divide it out, but using the Remainder Theorem is faster. From the Remainder Theorem, we know that $R = P(-1) = (-1)^{35} + (-1)^{17} - 5 = -1 - 1 - 5 = -7$.

7. to graph the polynomial P(x);

Consider polynomials of maximum degree 4 at this point. Points on the graph within a specified domain can be determined using synthetic division and the Remainder Theorem.

Example:

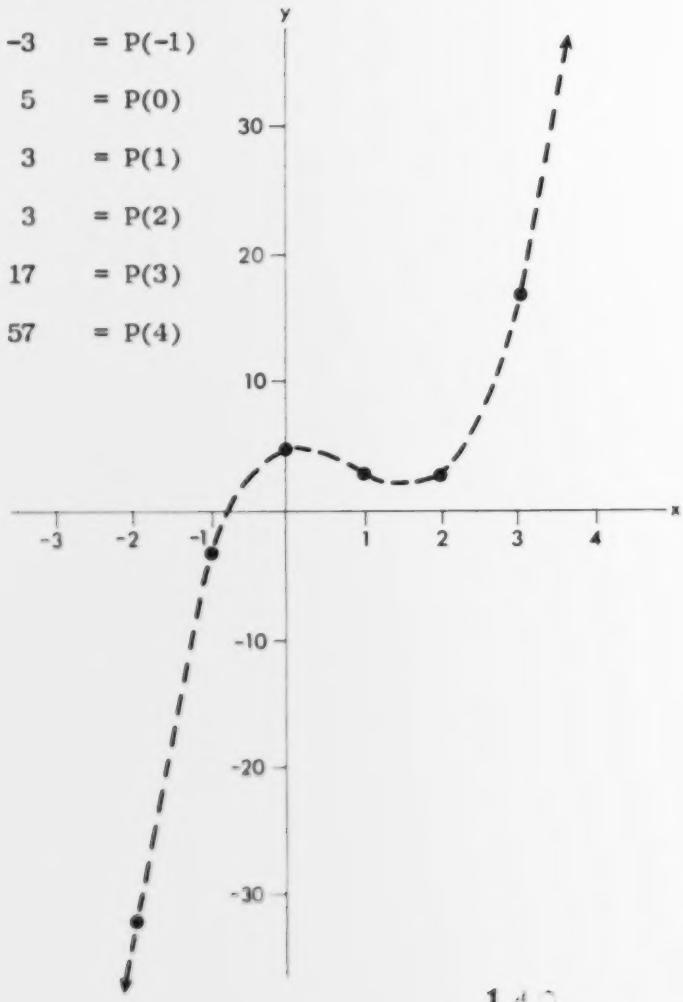
Graph $P(x) = 2x^3 - 5x^3 + x + 5$, $-2 < x < 5$

Using synthetic division, we can evaluate P(x) from $x = -2$ to $x = 4$ using selected values for x.

Set up a synthetic division table.

-2	2	-9	19	-33	= P(-2)
-1	2	-7	8	-3	= P(-1)
0	2	-5	1	5	= P(0)
1	2	-3	-2	3	= P(1)
2	2	-1	-1	3	= P(2)
3	2	1	4	17	= P(3)
4	2	3	13	57	= P(4)

Graph



OBJECTIVES	NOTES TO TEACHERS
8. to prove the Factor Theorem;	The Factor Theorem states: A polynomial P(x) has "r" as a zero iff it has x - r as a factor. Proof: $P(x) = (x - r)\,Q(x) + R$ From the Remainder theorem, R = P(r). Thus, $P(x) = (x - r)\,Q(x) + P(r)$. It can be seen that x - r is a factor of P(x) iff P(r) = 0; that is, if r is a zero of the polynomial P(x).
9. to find the zeroes of P(x) where $P(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0$ and $a_n = 1$;	The polynomials P(x) considered should be chosen with respect to the fact that the factor theorem is the students' primary tool. P(x) with integral coefficients and zeroes are advisable at the outset. Example: Find the zeroes of $P(x) = x^3 + 3x^2 - x - 3$. Using synthetic division 1 3 -1 -3 1 | 1 4 3 0 ∴ P(1) = 0 Therefore, x - 1 is a factor of $x^3 + 3x^2 - x - 3 = 0$. The other factor is $x^2 + 4x + 3$, which can be factored. By inspection $x^2 + 4x + 3 = (x + 1)(x + 3)$. Consequently, the zeroes of P(x) are 1, -1, and -3.

OBJECTIVES	NOTES TO TEACHERS
10. to state the Fundamental Theorem of Algebra;	Every polynomial $P(x)$ of degree $n > 1$ with real or complex coefficients has at least one real or complex zero.
11. to state the Complete Factorization Theorem;	The Complete Factorization Theorem states: Every polynomial $P(x)$ of degree $n > 1$, real or complex coefficients can be epxressed as a product of n linear factors, not necessarily distinct. Example: $P(x) = x^4 - 7x^2 + 4x + 20$ has 4 zeroes. Teachers may wish to introduce the term "zero of multiplicity." $P(x) = x^4 - 7x^2 - 4x + 20$ has 2 zeroes whose value is -2. Then, -2 is a zero of multiplicity 2.
12. to state the Complex Zeroes Theorem;	The Complex Zeroes Theorem states: If a polynomial with real coefficients has non-real complex zeroes, they exist in conjugate pairs. This can be illustrated by looking at the solution of $P(x) = x^2 - 6x + 10$. From the quadratic formula, $x = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm i$ Therefore, the zeroes of $P(x)$ are $3 + i$ and $3 - i$ which are conjugate complex numbers. A direct consequence of this is that a polynomial of odd degree always has at least one real zero.

OBJECTIVES	NOTES TO TEACHERS

Example:

Given the equation $x^3 - 4x^2 - 3x + 3 = 0$, what are the possible combinations of real and non-real complex solutions?

13. to state Descartes' Rule of Signs;

Descartes' Rule of Signs states:

Given a polynomial P(x) with real coefficients:

a) Positive real zeroes: The number of positive real zeroes of P(x) is never greater than the number of variations in sign in P(x) and, if less, then always by an even number.

b) Negative real zeroes: The number of negative real zeroes of P(x) is never greater than the number of variations in sign in P(-x) and, if less, then always by an even number.

Example of variations in sign

$$P(x) = \overset{1}{\frown} 4x^4 - \overset{1}{\frown} 5x^3 + \overset{1}{\frown} 2x^2 - 3x - 5$$

gives 3 variations in sign, and P(x) has either 3 or 1 positive real root. Similarly, the possible number of negative real roots can be found by examining the variations in sign of P(-x).

$$P(-x) = 4x^4 + 5x^3 + 2x^2 + 3x - 5$$

which produces 1 variation in sign and P(x) has 1 negative real root.

OBJECTIVES	NOTES TO TEACHERS

Conclusions: P(x) has either

a) 3 positive and 1 negative real roots, or
b) 1 positive and 1 negative real root.

14. to state the Boundary Theorem;

The Boundary Theorem states:

Given a polynomial P(x) with real coefficients, degree $n > 1$, and the coefficient of the nth degree term positive. Let P(x) be divided by $x - r$ using synthetic division.

1. Upper bound: If $r > 0$ and all numbers in the quotient row of the synthetic division are non-negative, then r is an upper bound of the zeroes of P(x).

2. Lower bound: If $r < 0$ and all numbers in the quotient row of the synthetic division alternate in sign, then r is a lower bound of the zeroes of P(x).

In the lower bound test, if 0 appears in one or more than one place in the quotient row, the sign preceding it can be taken as positive or negative.

Example

Find the smallest positive integer and the largest positive integer that are upper and lower bounds for the real zeroes of

$$P(x) = x^3 - x^2 - 6x + 6$$

Solution

Begin by testing $r = 1, 2, 3, \ldots$ until the quotient row turns non-negative.

Then, test $r = -1, -2, -3, \ldots$ until the quotient row alternates in sign:

OBJECTIVES	NOTES TO TEACHERS

	1	-1	-6	6	
1	1	0	-6	0	
2	1	1	-4	-2	
3	1	2	0	6	∴ 3 is an upper bound
-1	1	-2	-4	10	
-2	1	-3	0	6	
-3	1	-4	6	-12	∴ -3 is a lower bound

15. to state the Location Theorem;

The Location Theorem states:

If P(x) is a polynomial with real coefficients and P(a) and P(b) are of opposite signs, then there is at least one real zero between "a" and "b."

Example

Consider $P(x) = 2x^4 - 3x^3 - 3x - 4$.

Show that P(x) has at least one real root between 2 and 3.

P(2) = -2
P(3) = 68 using synthetic division.

Therefore, P(x) has at least one real zero between 2 and 3.

OBJECTIVES	NOTES TO TEACHERS
16. to state the Rational Zero Theorem;	The Rational Zero Theorem states: If the rational number $\frac{b}{c}$, (b and c are integers), in lowest terms, is a zero of the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, with integral coefficients, then b must be a factor of a_0, and c must be a factor of a_n. Example: For $P(x) = 3x^4 - 8x^3 - 6x^2 + 17x + 6$, list all possible rational zeroes. Solution: The factors of $a_0 = \pm 1, \pm 2, \pm 3, \pm 6$ and factors of $a_n = \pm 1, \pm 3$ Therefore, the possible zeroes are $\pm 1/3, \pm 2/3, \pm 1, \pm 2, \pm 3, \pm 6$
17. to find the rational zeroes for P(x), if they exist;	Polynomials of degree $n > 3$ with integral coefficients are the primary concern here. A strategy for finding the rational zeroes should be planned for the students: i) Determine all possible rational zeroes using the rational zero theorem. ii) Determine the possible number of positive and negative real zeroes using Descartes' Rule of Signs. iii) Test the possible rational zeroes from the list in (i). If a rational zero "r" is found, write $P(x) = Q(x)(x - r)$ and proceed to find the rational zeroes of Q(x). If Q(x) is of degree $n > 3$, return to step 1. However, if Q(x) is a quadratic, find all zeroes using standard methods for solving quadratic equations.

OBJECTIVES	NOTES TO TEACHERS
18. to find all the real zeroes of polynomial P(x);	Plan a strategy for finding all real zeroes: i) a) Find the rational zeroes using the strategy planned in objective 17. b) Repeat until all rational zeroes are determined. ii) Write the reduced polynomial Q(x) that results. iii) Use Descartes' Rule of Signs to determine the possible number of positive and negative real zeroes in Q(x). iv) Use a synthetic division table with Q(x) to: a) locate lower and upper bounds for irrational zeroes; b) isolate irrational zeroes, if possible, between successive integers by observing sign change in Q(x). v) Approximate irrational zeroes (to one decimal place) using a calculator.
19. to find all zeroes of polynomial P(x);	Two examples are illustrated. One has rational and non-real complex zeroes, while the other has rational, irrational and non-real complex zeroes. A calculator is most useful in performing the calculations. Example 1: Find all the zeroes of polynomial $P(x) = 2x^3 - 3x^2 + 3x - 1$ Solution Find all the rational zeroes using synthetic division:

	2	-3	3	-1	
1	2	-1	2	1	= P(1)
-1	2	-5	8	-9	= P(-1)
$\frac{1}{2}$	2	-2	2	0	= P($\frac{1}{2}$)

Therefore is a rational zero.

So that, $P(x) = (x - \frac{1}{2})(2x^2 - 2x + 2)$

$= 2(x - \frac{1}{2})(x^2 - x + 1)$.

Let $Q(x) = x^2 - x + 1$.

Using the quadratic formula

$$x = \frac{+1 \pm\sqrt{1 - 4(1)(1)}}{2}$$

$$= \frac{1 \pm\sqrt{-3}}{2}$$

$$= \frac{1}{2} \pm\frac{\sqrt{3}}{2}i$$

And the zeroes of P(x) are $\frac{1}{2}$, $\frac{1}{2} \pm\frac{\sqrt{3}}{2}i$

Example 2:

Find all the zeroes of polynomial

$$P(x) = x^4 - x^3 + 10x^2 - 28x + 18.$$

Using synthetic division

	1	-1	10	-28	18
1	1	0	10	-18	0

$\therefore P(1) = 0$

So that $P(x) = (x - 1)(x^3 + 10x - 18)$

Let $Q(x) = x^3 + 10x - 18$.

$Q(x) = x^3 + 10x - 18$ has one variation in sign

$\therefore Q(x)$ has one positive real zero.

$Q(-x) = -x^3 - 10x - 18$ has no variation in sign.

$\therefore Q(x)$ has no negative real zeroes.

Using either synthetic division or substitution:

$Q(1) = -7$	$Q(-1) = -29$
$Q(2) = 10$	$Q(-2) = -46$
$Q(3) = 39$	$Q(-3) = -75$

Therefore, the positive real zero is irrational and must lie between 1 and 2 (Location Theorem).

Using a calculator: $Q(1.5) = 0.375$
$Q(1.4) = -1.256$
$Q(1.6) = 2.096$

Therefore, using one decimal place accuracy, 1.5 is the positive real zero of $Q(x)$.

	1	0	10	−18
1.5	1	1.5	12.25	0.375

So that $Q(x) = (x - 1.5)(x^2 + 1.5x + 12.25)$.

Let $H(x) = x^2 + 1.5x + 12.25$.

Using the quadratic formula

$$x = \frac{-1.5 \pm \sqrt{(1.5)^2 - (4)(1)(12.25)}}{2}$$

$$= \frac{-1.5 \pm \sqrt{-46.75}}{2}$$

$$= -0.75 \pm 3.4\ i$$

The zeroes of $H(x)$ are: $-0.75 \pm 3.4\ i$.

The zeroes of $P(x)$ are: 1, 1.5, $-0.75 \pm 3.4\ i$ (some are approximated).

OBJECTIVES	NOTES TO TEACHERS
20. to solve word problems involving polynomial equations.	Problems can be found in most textbooks and include volume and digit questions. Example: A rectangular box has dimensions of 1 m by 2 m. If each side is increased by the same amount, how much should this increase be in order to triple the volume of the box?

TOPIC VI: MATRICES AND SYSTEMS OF LINEAR EQUATIONS (1 MODULE)

INTRODUCTION

A manufacturer of computer modems might want to organize some of the data pertaining to production so it can be processed easily on a computer. Suppose three models are being made, as follows:

	Model A	Model B	Model C
Number of Chips	13	18	20
Number of Switches	2	3	4

The numbers can be arranged as $\begin{bmatrix} 13 & 18 & 20 \\ 2 & 2 & 4 \end{bmatrix}$ in a parts-per-modem matrix. In addition, orders received for modems can be arranged in a matrix, as follows (the name is attributed to J.J. Sylvester, who used it to describe an oblong arrangement of terms of "m" lines and "n" columns):

	January	February
Model A	12	6
Model B	24	12
Model C	12	9

The parts-per-month matrix is $\begin{bmatrix} 12 & 6 \\ 24 & 12 \\ 12 & 9 \end{bmatrix}$

Now, if the computer can work with matrices, the total number of parts needed in these two months is:

$$\begin{bmatrix} 13 & 18 & 20 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 12 & 6 \\ 24 & 12 \\ 12 & 9 \end{bmatrix} = \begin{matrix} & J & F \\ \text{Chips} & 828 & 474 \\ \text{Switches} & 144 & 84 \end{matrix}$$

Thus, in January 828 chips are needed.
Thus, in February 474 chips are needed.
in January 144 switches are needed.
in February 84 switches are needed.

The fact that matrices can be manipulated using simple consistent rules makes solving problems like this relatively painless. This topic is intended to enable students to master basic operations on matrices and eventually to use matrix manipulation to solve systems of linear equations. Linear systems of two variables in two unknowns and linear systems of three variable sin three unknowns will be discussed. Other systems work similarly.

OBJECTIVES	NOTES TO TEACHERS
The student should be able:	
1. to define the term "matrix";	Matrix: a rectangular array of real numbers e.g., $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$
2. to state the dimensions of a matrix;	State first the number of rows, second, the number of columns. Example 1: $[1, 2, 3]$ is a 1 by 3 matrix and is sometimes called a row vector. Example 2: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is called a 3 by 1 matrix or a column vector.
3. to find entries in a matrix given subscripts;	Example: $\begin{bmatrix} 7 & 2 \\ 1 & 3 \end{bmatrix}$, find m_{21} Solution $m_{21} = 1$

OBJECTIVES	NOTES TO TEACHERS
	Order For a Square Matrix
	- A two-by-two matrix is called a square matrix of order 2. - A three-by-three matrix is called a square matrix of order 3.
4. to determine conditions under which matrices are said to be equal;	Example: Under which conditions are $A = \begin{bmatrix} 2x + 3y \\ 3x - y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ equal?
	Solution: When $2x + 3y = 7$ and $3x - y = 2$ They are equal when $x = \frac{13}{11}$ and $y = \frac{17}{11}$.
5. to define addition for matrices;	Two matrices of the same order are added as follows: If A has entries a_{ij} and B has entries b_{ij}, their sum is matrix C with entries c_{ij}. $c_{ij} = a_{ij} + b_{ij}$. C is of same order as A and B.
6. to define the identity element for addition of matrices of a given order;	Identity element: A matrix of the given dimensions with each entry a zero and is called the zero matrix.
7. to show that addition of matrices is commutative;	Addition is possible only with matrices of the same order.
8. to show that addition of matrices is associative;	
9. to state the negative or the additive inverse of a given matrix;	

OBJECTIVES	NOTES TO TEACHERS
10. to define subtraction for matrices;	$A - B = A + (-B)$
11. to multiply a matrix by a number;	If x and y are numbers and A and B are matrices: a) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $xA = \begin{bmatrix} xa_{11} & xa_{12} \\ xa_{21} & xa_{22} \end{bmatrix}$ b) $x(yA) = xy(A)$ c) $(x + y)A = xA + yA$ d) $(-1)A = -A$ e) $x(A + B) = xA + xB$ f) $x\underline{0} = \underline{0}$ where $\underline{0}$ is the zero matrix g) $0A = 0$ The product of a number and a matrix is always a matrix.
12. to multiply a matrix by a matrix;	Refer to the chips parts problem at the beginning of this topic. Show that common sense indicates that the four numbers in the product matrix are correct. Then, discuss row by column multiplication and show that matrices can be multiplied only if the number of columns in the left matrix equals the number of rows in the right matrix.

OBJECTIVES	NOTES TO TEACHERS

13. to show that multiplication of matrices is not generally commutative;

If $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Hamilton thought this fact was important enough to carve it into a bridge stone.

Note: Filling a pool and diving in are also not commutative.

OBJECTIVES	NOTES TO TEACHERS
14. to observe that the product of two non-zero matrices may be the zero matrix;	Example: $\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ This leads to the fact that whereas with real numbers, if $ab = ac$ and $a \neq 0$, then $b = c$ (cancellative law), we have $AB = AC$, with $A \neq 0$ and $B \neq C$. Note: The existence is possible of infinitely many matrices A such that $A^2 = B$.
15. to show that multiplication is associative;	
16. to show that multiplication is Left Distributive;	
17. to show that multiplication of matrices is Right Distributive;	
18. to show that in general $A(B+C) \neq (B+C)A$;	
19. to define the unit matrix of order n;	

OBJECTIVES	NOTES TO TEACHERS

20. to show that IA = A = AI;

Example:

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$$

Notes: 1) I is a two-by-two in one case and a three-by-three in the other. This is to conform with the rule of multiplying matrices.
2) The appropriate unit matrix m ust be used in each case.

21. to solve a linear system by using row operations (restrict to square systems);

Note: Row operations permitted are:

1. Multiply an equation by a non-zero number;
2. Add one equation to another; and
3. Interchange equations.

I
$$\begin{aligned} x - y + z &= -2 && (1)\\ x - 2y - 2z &= -1 && (2)\\ 2x + y + 3z &= 1 && (3)\end{aligned}$$

II
$$\begin{aligned} x - y + z &= -2 && (1)\\ 0 - y - 3z &= 1 && (2) + -1(1)\\ 0 + 3y + z &= 5 && (3) + -2(1)\end{aligned}$$

OBJECTIVES	NOTES TO TEACHERS

III
$$\begin{aligned} x - y + z &= -2 && (1) \\ 0 + y + 3z &= -1 && (1) + -1(2) \\ 0 + 0 - 8z &= 8 && (3) + 3(2) \end{aligned}$$

IV
$$\begin{aligned} x - y + 0 &= -1 && \tfrac{1}{8}(3) + (1) \\ 0 + y + 0 &= 2 && (2) + \tfrac{3}{8}(3) \\ 0 + 0 + z &= -1 && \tfrac{1}{8}(3) \end{aligned}$$

V
$$\begin{aligned} x + 0 + 0 &= 1 && (1) + (2) \\ 0 + y + 0 &= 2 && (2) \\ 0 + 0 + z &= -1 && (30 \end{aligned}$$

Thus $x = 1 \qquad y = 2 \qquad z = -1$

Since each operation is reversible, the values satisfying (V) also satisfy (I) uniquely. Each of the five systems is equivalent.

22. to formulate a system of equations in terms of matrices;

The example in objective 21 can be written as follows:

I
$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

II
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

OBJECTIVES	NOTES TO TEACHERS

III $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 8 \end{bmatrix}$

IV $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

V $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

23. to find the inverse (if it exists) for a square matrix, using elementary row operations;

Definitions of Inverse For A Square Matrix

Square matrices P and Q, with $PQ = QP = I$ are inverses. We say $Q = P^{-1}$, i.e., Q is the inverse of P. Likewise $P = Q^{-1}$.

Example:

We had

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$A \qquad X = B$$

We could also say

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$A \quad X = I \quad B$$

Note: If there is a matrix A^{-1} such that $A^{-1}A = I$, we can multiply both of this equation by A^{-1} as shown below.

$$\begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$A \quad X \quad A^{-1} \quad I \quad B$$

This would reduce to $IX = A^{-1}IB$.

In short, $X = A^{-1}B$

Row operations done simultaneously on A and I to change A to I will change I to A^{-1}.

The five steps show below are the same five steps as in the previous example:

$$I \quad \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & -2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

OBJECTIVES	NOTES TO TEACHERS

II $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -1 & 1 & 0 \\ 0 & 3 & 1 & -2 & 0 & 1 \end{array}\right]$

III $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & -1 & 0 \\ 0 & 0 & -8 & -5 & 3 & 1 \end{array}\right]$

IV $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & 1 & 0 & \frac{-7}{8} & \frac{1}{8} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{8} & \frac{-3}{8} & \frac{-1}{8} \end{array}\right]$

V $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-4}{8} & \frac{4}{8} & \frac{4}{8} \\ 0 & 1 & 0 & \frac{-7}{8} & \frac{1}{8} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{5}{8} & \frac{-3}{8} & \frac{-1}{8} \end{array}\right]$

Thus $A^{-1} =$

OBJECTIVES	NOTES TO TEACHERS

This is the inverse of A.

Note: Not all square matrices have inverses. Determinants will be used to show this in earlier courses.

Note: A^{-1} is equivalent to $\longrightarrow \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

24. to solve matrix equations using inverses;

$$\begin{bmatrix} \frac{-4}{8} & \frac{4}{8} & \frac{4}{8} \\ \frac{-7}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{5}{8} & \frac{-3}{8} & \frac{-1}{8} \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Note: If a row of zeroes appears at some point in the left matrix, either an infinite number of solutions will be found or there will be no solutions. The notion of determinants has not been developed. While historically important, determinants have been outdated due to the speed and power of computer programs which can handle large matrices and solve problems involving them faster with row operations than with determinants. The most efficient method is called the LU Method and should be studied in a first year university course in Linear Algebra.

25. to solve systems of linear equations using inverses.

TOPIC VII: THREE DIMENSIONAL GEOMETRY (1 MODULE)

INTRODUCTORY EXAMPLES:

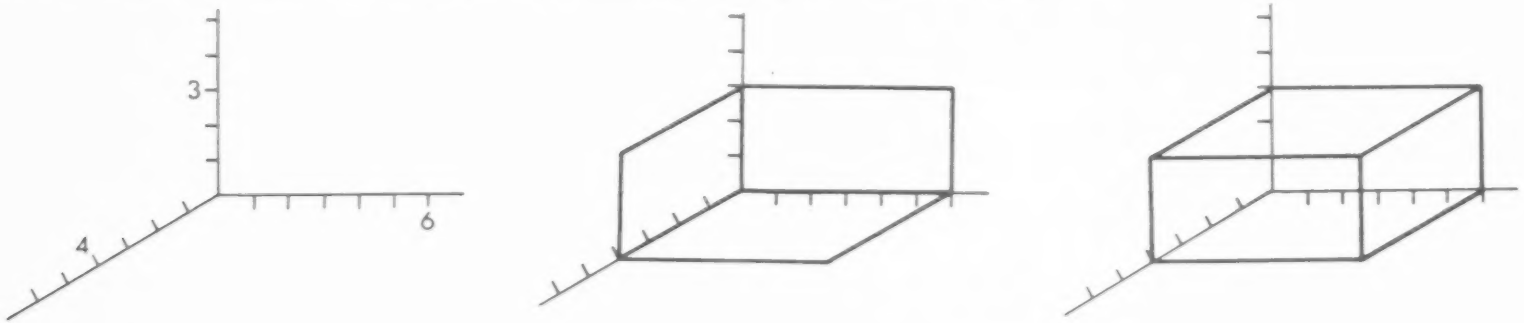
Example 1:

We draw the coordinate box for the point (4,6,3) in one of the several possible ways. The steps, shown in order below, are:

1. Mark the three coordinate points on the axes.

2. Through each marked point, draw the two lines parallel to the other two axes, thus obtaining three rectangles, one in each coordinate plane. These are the three faces of the coordinate box that have the origin as a vertex.

3. Each of the three rectangles just drawn has one new vertex. Through it, draw a line representing a perpendicular to its coordinate plane; that is, a line parallel to the axis not in its coordinate plane. These three lines will meet at the point representing the space point (4,6,3).

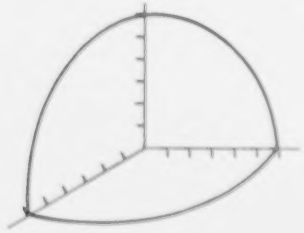


Example 2:

The graph of $x^2 + y^2 + z^2 = 36$ is the sphere about the origin of radius 6. For, taking square roots and replacing x by x - 0, etc., we get the equivalent equation

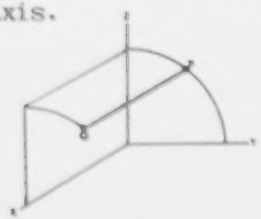
$$\sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = 6,$$

which says that the distance from (0,0,0) to (x,y,z) is 6, by the distance formula. The graph consists of all such points (x,y,z) and is thus the sphere. Below is a drawing of the part of the sphere in the first octant.

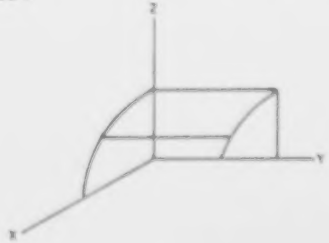


Example 3:

If a point P lies on the graph of $2z = 4 - y^2$, then so does any other point Q having the same y- and z-coordinates. The graph thus contains the line through P parallel to the x-axis.



A piece of the graph in the first octant is shown above. Every plane $x = a$ is a yz-plane which intersects the surface in the same parabola $2z = 4 - y^2$. Similarly, the graph $2z = 4 - x^2$. Similarly, the graph of $2z = 4 - x^2$ is a "parabolic cylinder" parallel to the y-axis, as shown below at the left. These two parabolic cylinders, together with the coordinate planes, bound a solid figure in the first octant, as sketched at the right. It has square cross sections parallel to the xy-plane.



Note: When computers are available, they should be used to produce the graphs.

OBJECTIVES	NOTES TO TEACHERS

The student should be able:

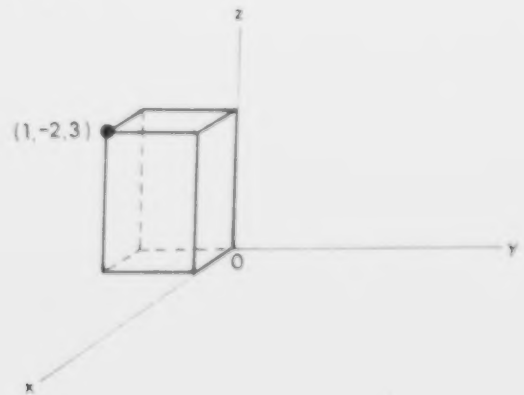
1. to demonstrate an understanding of the terms used in discussing a rectangular coordinate system in space;

The student should understand the terms: coordinate planes, coordinates of a point, octant, first octant.

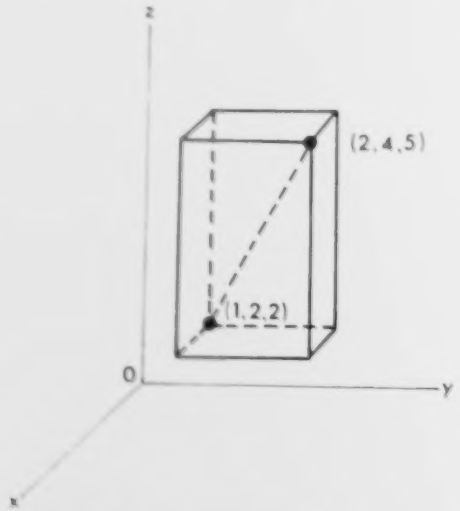
2. to draw a sketch of the coordinate axes and to represent a point with given coordinates on such a sketch;

Example:

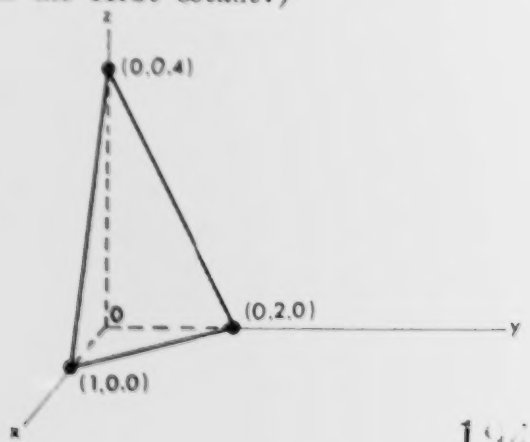
Represent the point (1, -2, 3) on a sketch of a rectangular coordinate system.



OBJECTIVES	NOTES TO TEACHERS
3. to represent certain figures in space by plane drawings;	1. Using a sketch of a coordinate system, draw a rectangular box with sides parallel to the coordinate planes and having (1,2,2) and (2,4,5) at diagonally opposite corners. Indicate the line segment joining (1,2,2) and (2,4,5).



2. Sketch the plane containing the points (1,0,0), (0,2,0), (0,0,4). (Show only the portion in the first octant.)

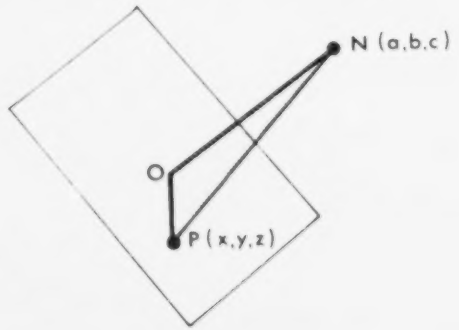


OBJECTIVES	NOTES TO TEACHERS
4. to prove the distance formula;	The distance between two points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) is given by the formula: $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
5. to demonstrate an understanding of the term "equation of a surface";	An equation of a surface S must satisfy the following two conditions: 1. The co-ordinate of each point on S satisfy the equation. 2. The co-ordinates of any point not on S do not satisfy the equation.
6. to find an equation of a sphere given the centre and radius;	Example: Find an equation of the sphere with centre at the origin and radius 3. Solution: Using the distance formula, we see that an equation of the sphere is $x^2 + y^2 + z^2 = 9$.
	ENRICHMENT Show that $x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$ is an equation of a sphere. What is the centre and the radius of the sphere?
7. to show that an equation of a plane which passes	Let N be the point (a,b,c). Let P be the point (x,y,z). P is on the given plane if and only if $\angle PON = 90°$; that is, if and only if $(NP)^2 = (OP)^2 +$

OBJECTIVES	NOTES TO TEACHERS

through the origin and is perpendicular to the line segment joining the origin and (a,b,c) is ax + by + cz = 0;

$(ON)^2$. This equation is equivalent to $(x-a)^2+(y-b)^2+(z-c)^2 = x^2+y^2+z^2+a^2+b^2+c^2$, which is equivalent to ax + by + cz = 0.



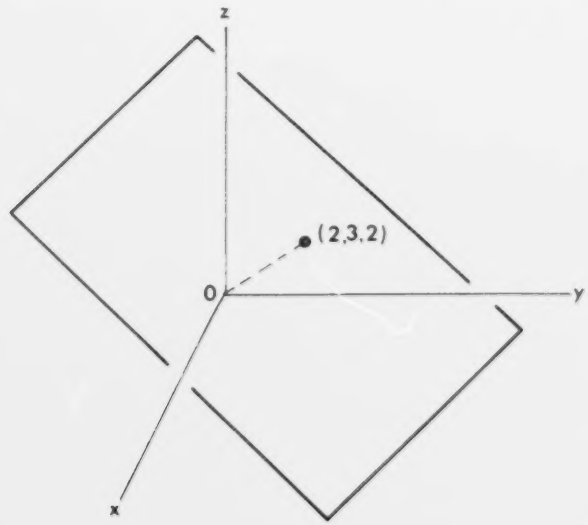
Example:

Find an equation of the plane which passes through the origin and is perpendicular to the line segment joining the origin and (2,3,2).

Solution:

2x + 3y + 2z = 0.

8. to sketch a plane such as 2x + 3y + 2z = 0;



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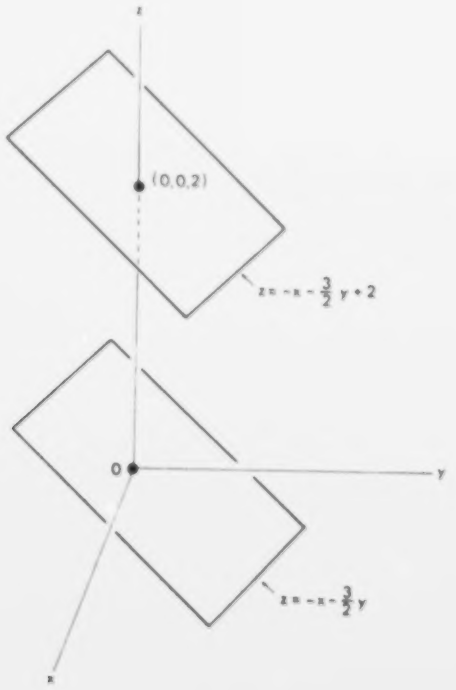
OBJECTIVES	NOTES TO TEACHERS

9. to show that $2x + 3y + 2z - 4 = 0$ is an equation of a plane parallel to $2x + 3y + 2z = 0$;



The equation can be written in the form:

$$z = -x - \frac{3}{2}y + 2$$

$$z = -x - \frac{3}{2}y$$

We see that the first surface may be obtained by translating the second surface upwards by w units. Thus the surface $2x + 3y + 2z - 4 = 0$ is a plane parallel to the plane $2x + 3y + 2z = 0$.

10. to recognize that any equation of the form $ax + by + cz + d = 0$ is an equation of a plane (provided a, b and c are not all zero) and that any plane has such an equation;

The teacher may deal with this objective using the method described in Objective 9.

OBJECTIVES	NOTES TO TEACHERS
11. to find the equation of a plane satisfying certain conditions;	Example 1: Find an equation of the plane passing through (1,2,3) and perpendicular to the line segment joining (0,0,0) and (4,5,6). Solution: The plane has an equation of the form $4x + 5y + 6z + d = 0$. Since (1,2,3) is on the plane, its co-ordinates satisfy this equation, so $4(1) + 5(2) + 6(3) + d = 0$; $\therefore d = -32$. Thus an equation of the plane is: $$4x + 5y + 6z - 32 = 0$$ Example 2: Find an equation of the plane passing through (2,0,0), (0,3,0) and (0,0,4). Solution: The plane has an equation of the form $ax + by + cz + d = 0$. Dividing by d we can write this equation in the form $$\frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$$ (since the plane clearly does not pass through 0, we know $d \neq 0$) or $Ax + By + Cz + 1 = 0$ (writing $A = \frac{a}{d}$, $B = \frac{b}{d}$, $C = \frac{c}{d}$). Since (2,0,0) is on the plane, its coordinates satisfy this equation, so $A(2) + B(0) + C(0) + 1 + 0$, $\therefore A = -\frac{1}{2}$.

OBJECTIVES	NOTES TO TEACHERS

Similarly, $A(0) + B(3) + C(0) + 1 = 0$, so $B = -\frac{1}{3}$

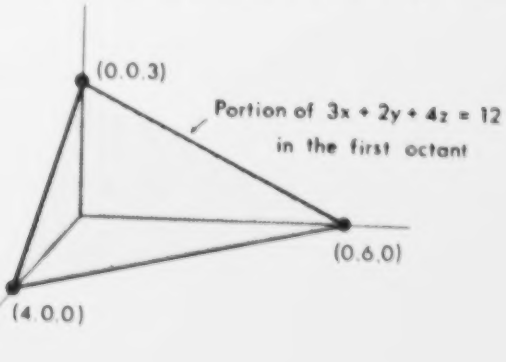
and, $A(0) + B(0) + C(4) + 1 = 0$, so $C = -\frac{1}{4}$

$\therefore -\frac{1}{2}x - \frac{1}{3}y - \frac{1}{4}z + 1 = 0$ is an equation of the given plane.

12. to sketch a plane whose equation is given;

Example 1:

Sketch the plane whose equation is $3x + 2y + 4z = 12$.

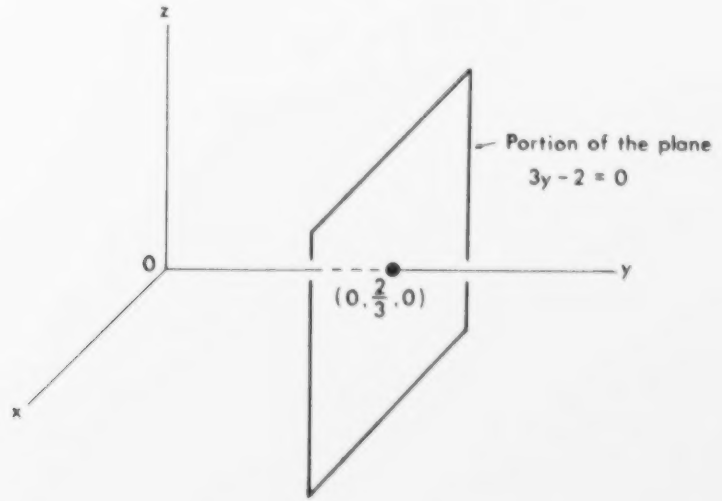


Example 2:

Sketch the plane whose equation is $x + 2y - 4 = 0$.

Example 3:

Sketch the plane whose equation is $3y - 2 = 0$.



(In sketching these planes it is useful to find where the planes meet the coordinate axes.)

Example 4:

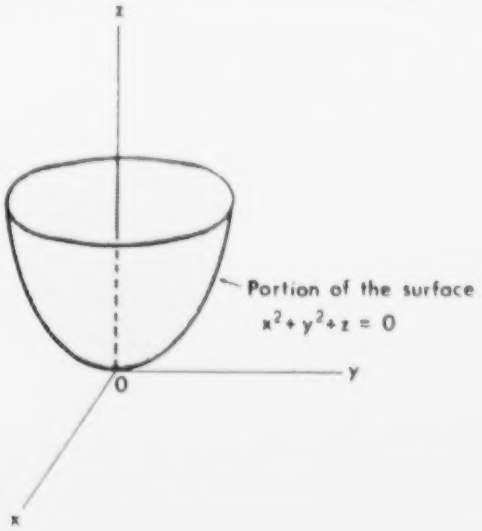
Sketch the plane whose equation is $x = y$.

(See the sketch in Objective 16.)

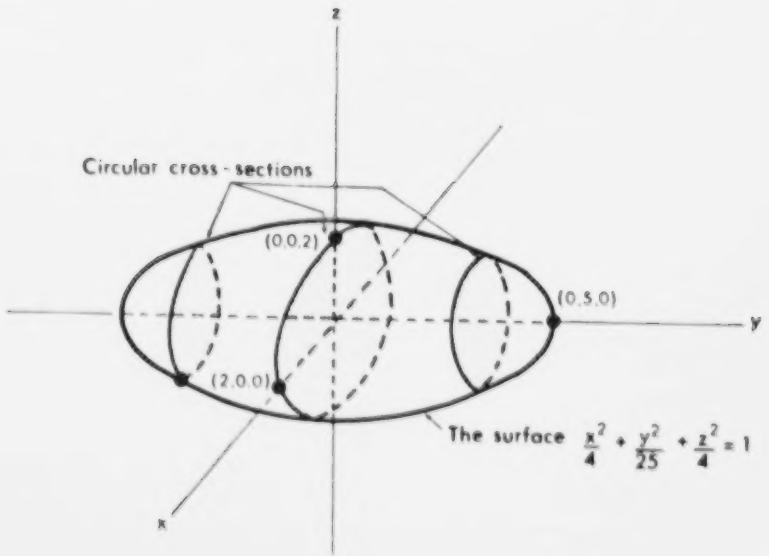
OBJECTIVES	NOTES TO TEACHERS

13. to sketch the surfaces whose equations are:

(a) $x^2 + y^2 - z = 0$,



(b) $\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{4} = 1$;



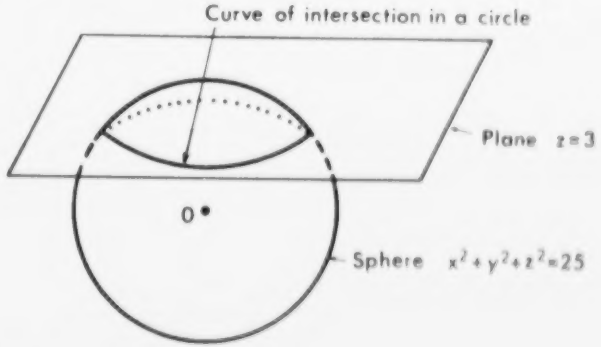
These surfaces are mentioned as examples which are neigher planes nor spheres and which are fairly easy to sketch. The terms "trace" and "intercept" may be introduced here.

OBJECTIVES	NOTES TO TEACHERS

14. to sketch the curve of intersection of two surfaces for some simple cases;

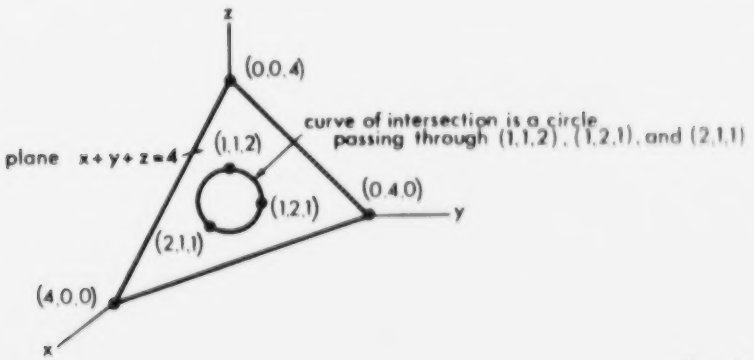
Example 1:

Sketch the curve of intersection of $x^2 + y^2 + z^2 = 25$ and $z = 3$.



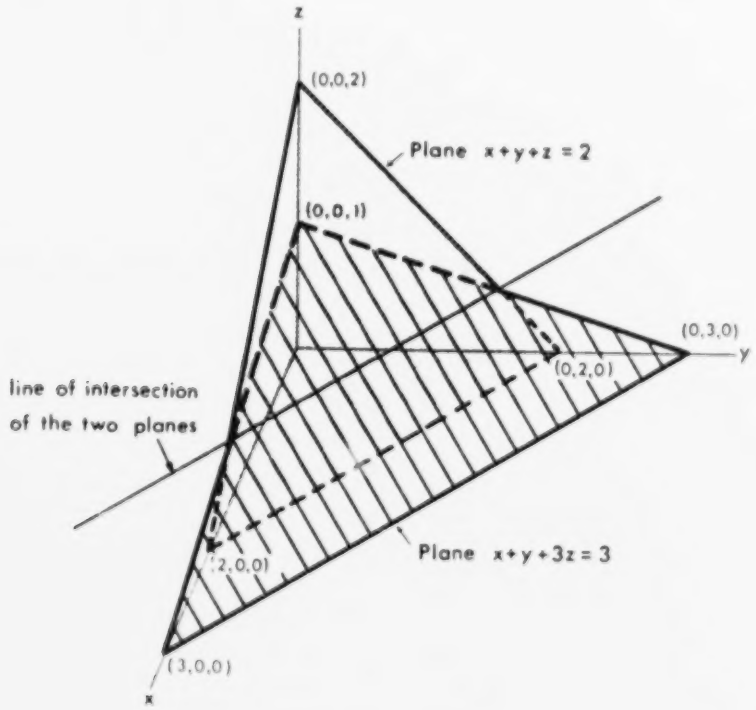
Example 2:

Sketch the curve of intersection of $x^2 + y^2 + z^2 = 6$ and $x + y + z = 4$.



Example 3:

Sketch the curve of intersection of $x + y + z = 2$ and $x + y + 3z = 3$.



OBJECTIVES	NOTES TO TEACHERS
15. to find two points on the line of intersection of two non-parallel planes;	Example: Find two points on the line of intersection of $x + y + z = 2$ and $2x + 3y + z = 3$. Solution: We find the point where the line meets the plane $x = 0$, by setting $x = 0$ in the two equations to obtain $y + 3 = 3$ and $3y + z = 3$. Solving simultaneously, we get $y = \frac{1}{2}$, $z = \frac{3}{2}$. Thus, the line meets the plane $x = 0$ at $(0, \frac{1}{2}, \frac{3}{2})$. We find the point where the line meets the plane $y = 0$ by setting $y = 0$ in the two equations to obtain $x + z = 2$ and $2x + z = 3$. Solving simultaneously, we get $x = 1$, $z = 1$. Thus, the line meets the plane $y = 0$ at $(1, 0, 1)$. Two points on the line are $(0, \frac{1}{2}, \frac{3}{2})$ and $(1, 0, 1)$.

OBJECTIVES	NOTES TO TEACHERS

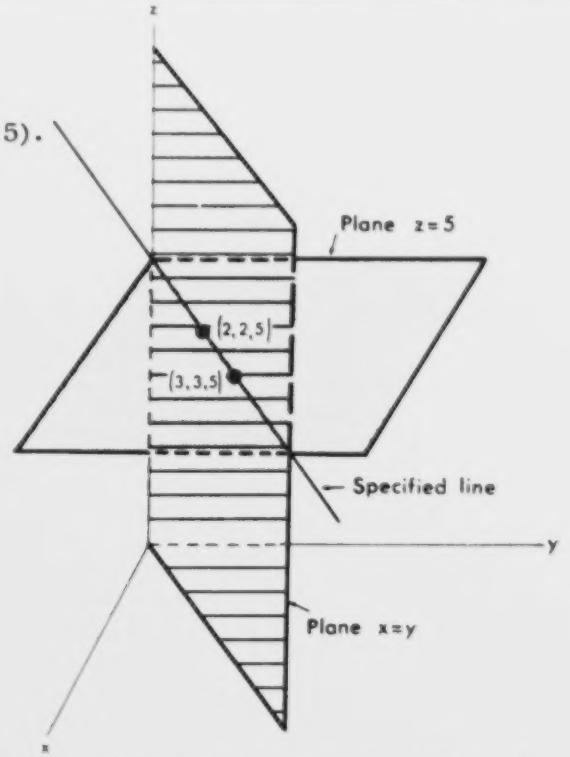
16. to find two planes whose intersection is a specified line (simple examples).

Example:

Find two planes whose intersection is the line joining (2, 2, 5) and (3, 3, 5).

Solution:

We notice that the z value of both points is 5.
Therefore, both points lie on the plane z = 5.
We also notice that for each point, the x value and the y value are equal.
Therefore, both points lie on the plane x = y.
The specified line may, therefore, be represented as the intersection of the planes x = y and z = 5.

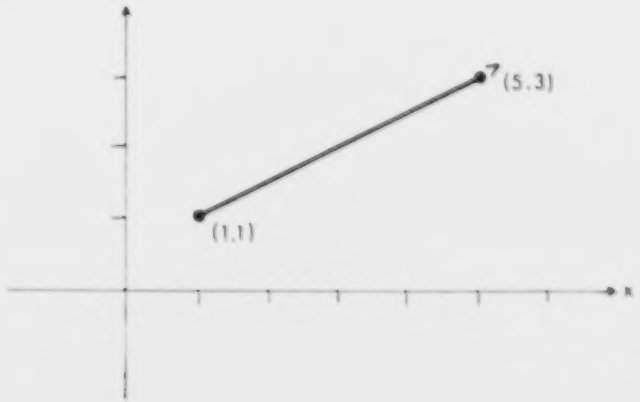


(This same line may be represented as the intersection of other pairs of planes.)

ENRICHMENT

Find two planes whose intersection is the line joining (2,0,1) and (0,3,4).

TOPIC VIII: VECTORS (1® MODULES)

OBJECTIVES	NOTES TO TEACHERS
The student should be able:	
1. to demonstrate an understanding of certain definitions, properties and notations associated with vectors;	Vector: A non-zero vector has two fundamental properties: a positive real number, called magnitude, and a direction in space. A vector may be denoted in several ways: $\overrightarrow{AB}$, V or $\overrightarrow{V}$. Equivalent Vectors: Two vectors are equivalent if they have the same direction and the same magnitude. Null Vector: The null vector is a vector whose magnitude is zero. It is denoted by 0.
2. to sketch vectors on a coordinate plane;	Example: 

OBJECTIVES	NOTES TO TEACHERS

3. to determine the magnitude of a vector in a plane;

Example:

$$v = \sqrt{(1-5)^2 + (1-3)^2} = \sqrt{16} + 4 = \sqrt{20} = 2\sqrt{5}$$

4. to demonstrate an understanding of the properties of multiplication of a vector by scalar;

If k is a real number and $\vec{V}$ is a non zero vector, then $k\vec{V}$ is a vector such that:

(a) $k\vec{V} = 0$ if $k = 0$.

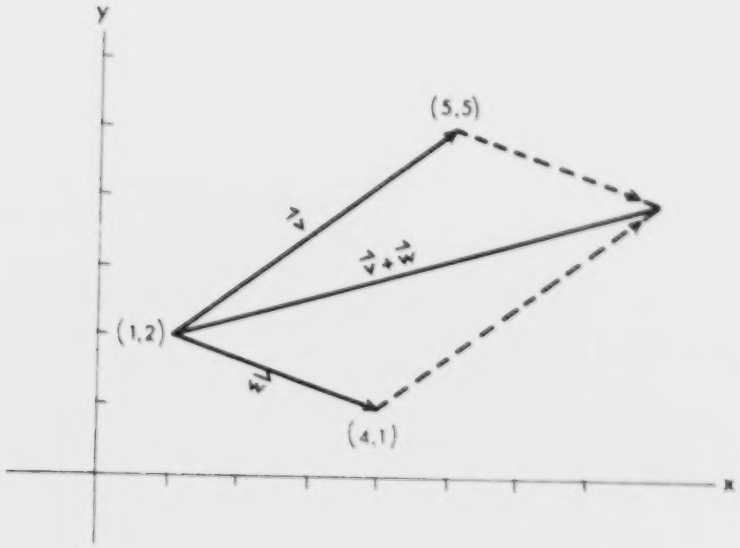
(b) $k\vec{V}$ ha the same direction as $\vec{V}$ if $k > 0$.

(c) $k\vec{V}$ has the opposite direction if $K < 0$.

(d) $k\vec{V}$ is the inverse of $\vec{V}$ if $k = -1$.

(e) The magnitude of $k\vec{V}$ is $[|k|]\,[|\vec{V}|]$.

5. to sketch the sum of difference of vectors;

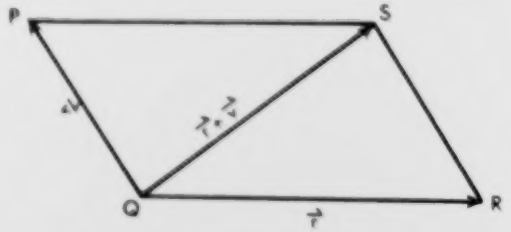


OBJECTIVES	NOTES TO TEACHERS

6. to solve problems using vector addition and subtraction, using the parallelogram rule;

Example:

Two vectors have a common initial point, as shown in diagram. If the vectors form an angle of 120° and $|\vec{r}| = 10$ and $|\vec{v}| = 6$, find $|\vec{r} + \vec{v}|$.



PS = 10, PQ = 6. $\angle QPS = 60°$ using the Cosine Law:

$QS^2 = PS^2 + PQ^2 - 2(PS)(PQ)\ \text{Cos}\ \angle QPS$

$= 100 + 36 - 2(10)(6)\ \text{Cos}\ 60°$

$= 76$

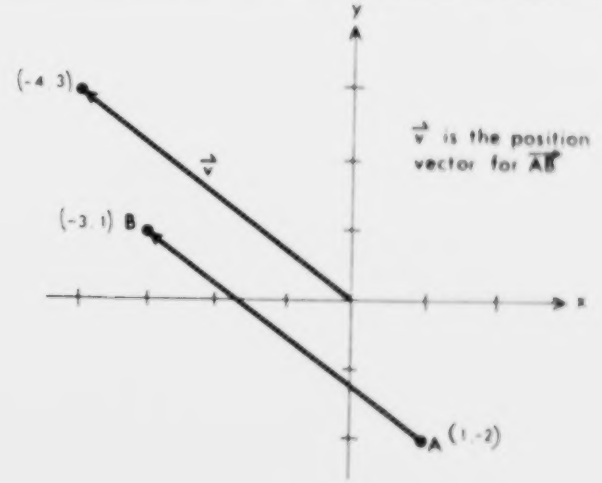
$\therefore QS = 2\sqrt{19}$

$\therefore |\vec{r} + \vec{v}| = 2\sqrt{19}$

7. to represent any vector in a plane by its position vector;

Example:

Represent the vector $\overrightarrow{AB}$ determined by the points A (1,-2) and B(-3,1) by a position vector. Give the coordinates of the terminal point of the position vector.

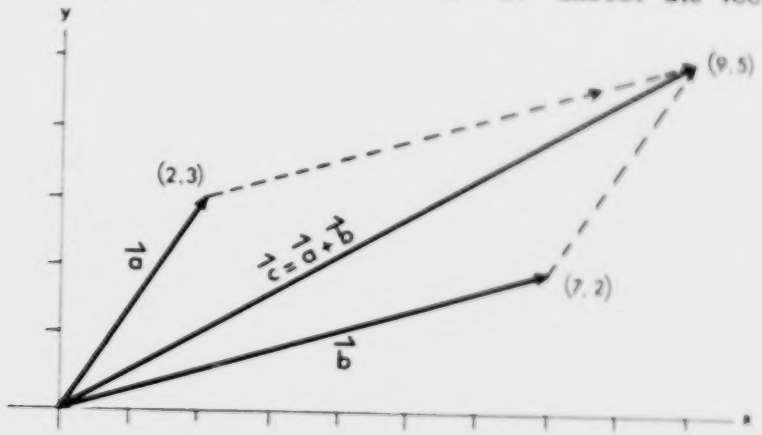


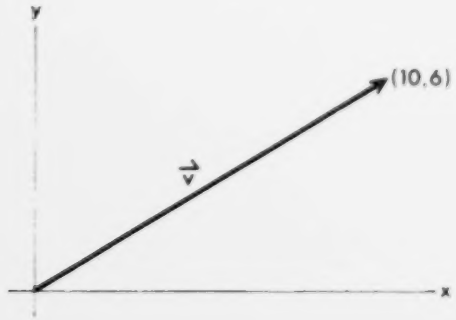
Note: A position vector is a vector whose initial point is at the origin.

8. to add and subtract position vectors by composition of coordinates;

Example:

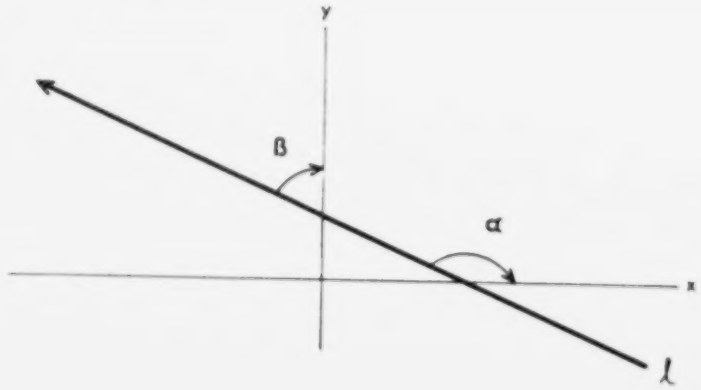
Vectors $\vec{a}$ and $\vec{b}$ have terminal points (2,3) and (7,2), respectively. Find the terminal point of $\vec{c}$ so that a + b = c. Sketch the vectors.



OBJECTIVES	NOTES TO TEACHERS
9. to demonstrate an understanding of the linear combination of vectors;	If $\vec{u}$ and $\vec{v}$ are two non-zero vectors which do not have the same or opposite directions, and $\vec{w}$ is any other vector in the same plane, there exist unique real numbers p and q such that: $\vec{w} = p\vec{u} + q\vec{v}$ Note: $\vec{u}$ and $\vec{v}$ are called "basis vectors."
10. to express any position vector in terms of its components;	$\vec{i} = (1,0)$ $\vec{j} = (0,1)$ Example: Express $\vec{v}$ in terms of its components. 
11. to add and subtract position vectors which have been expressed in terms of their components;	Example: If $\vec{r} = 3\vec{i} - 6\vec{j}$ and $\vec{v} = \vec{i} + 4\vec{j}$, find (a) $\vec{r} + \vec{v}$ and (b) $\vec{v} - \vec{r}$

OBJECTIVES	NOTES TO TEACHERS

12. to identify the direction angles of a line in a plane;



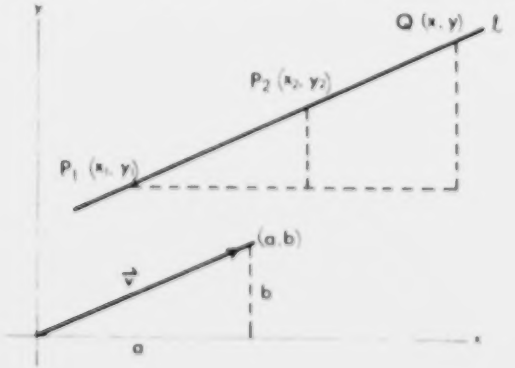
α and β are the direction angles. Cos α and Cos β are direction cosines of line ℓ.

13. to determine the direction cosines of a line in a plane;

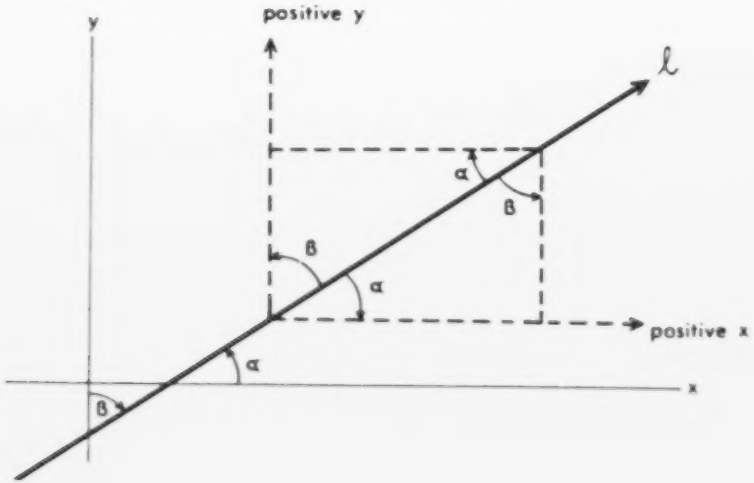
The line through P_1 and P_2 has $\frac{x - y_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$ (by similar triangles).

If $x_2 - x_1 = a$ and $y_2 - y_1 = b$. We get $\frac{x - x_1}{a} = \frac{y - y_1}{b}$

The ordered pair (a,b) is called a set of direction numbers for the line ℓ, and the line ℓ is parallel to the vector (a,b).



The line ℓ makes angles α and β with the direction of the positive x and positive y axes. These are called the direction angles and are determined by the direction cosines.



$$\text{Cos } \alpha = \frac{x_2 - x_1}{\left|\overrightarrow{P_1P_2}\right|}, \quad \text{Cos } \beta = \frac{y_2 \quad y_1}{\left|\overrightarrow{P_1P_2}\right|} \quad \text{where } \left|\overrightarrow{P_1P_2}\right| = \text{magnitude of } \overrightarrow{P_1P_2}$$

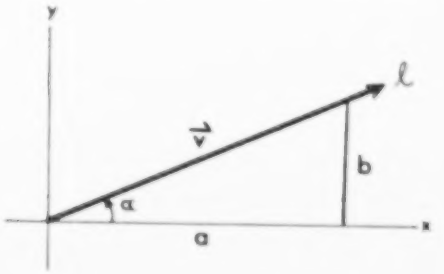
If line ℓ has direction numbers (a,b) then its direction cosines are

$$\text{Cos } \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \quad \text{Cos } \beta = \frac{b}{\sqrt{a^2 + b^2}}$$

Note: a and b control the sign of the cosine.

OBJECTIVES	NOTES TO TEACHERS

Since the vector $\vec{V} = (a,b)$ is shown and is parallel to line ℓ.



14. to find the magnitude of a vector which is expressed in terms of its components;

Example:

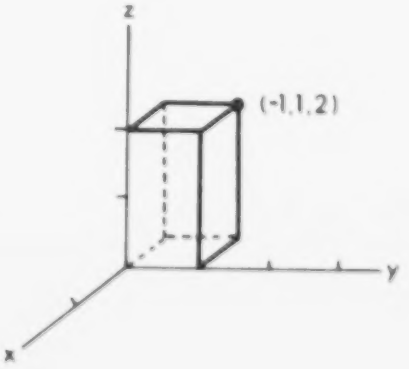
Find the magnitude of the vector $\vec{r} = 3\vec{i} - 4\vec{j}$.

Note: The vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are special unit vectors.

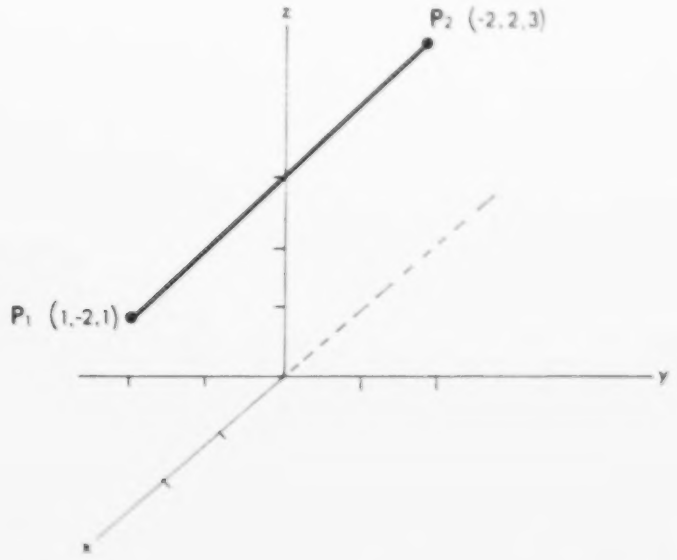
15. to demonstrate an understanding of certain definitions and axioms associated with three-dimensional geometry;

a) There is exactly one plane containing any three non-collinear points.
b) If two distinct planes intersect, they intersect in a line.
c) The line joining two distinct points in a plane lies in that plane.

16. to plot ordered triplets on a coordinate system;

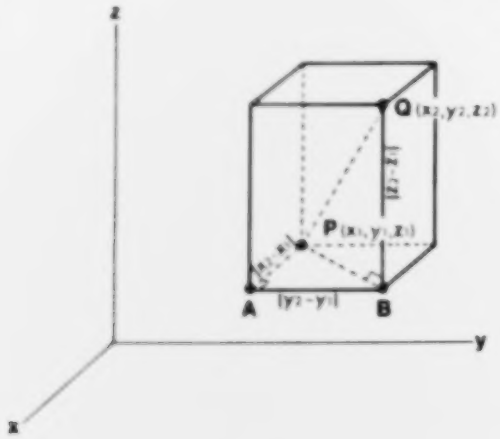


OBJECTIVES	NOTES TO TEACHERS
17. to represent a vector in 3-space on a plane diagram;	Example: If P_1 is the point (1,-2,1) and P_2 is the point (-2,2,3), indicate the vector P_1P_2 on a diagram of the coordinate system.



OBJECTIVES	NOTES TO TEACHERS
18. to derive the distance formula for 3-space;	This will only be necessary if it has not been done in Topic VII.

Prove: $d(P,Q) = PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$



Proof: See page 345 and 346 of Math 31, Elliott, et al., Holt, Rinehart and Winston, 1984.

OBJECTIVES	NOTES TO TEACHERS

19. to determine the magnitude of a vector in 3-space;

Example:

Find the length of the vector represented by the line segment from P(-1,2,5) to Q(-4,-3,1).

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(-4 - (-1)^2 + (-3 - 2)^2 + (1 - 5)^2}$$

$$= \sqrt{(-3)^2 + (-5)^2 + (-4)^2}$$

$$= \sqrt{9 + 25 + 16}$$

$$= \sqrt{50}$$

$$|PQ| = 5\sqrt{2}$$

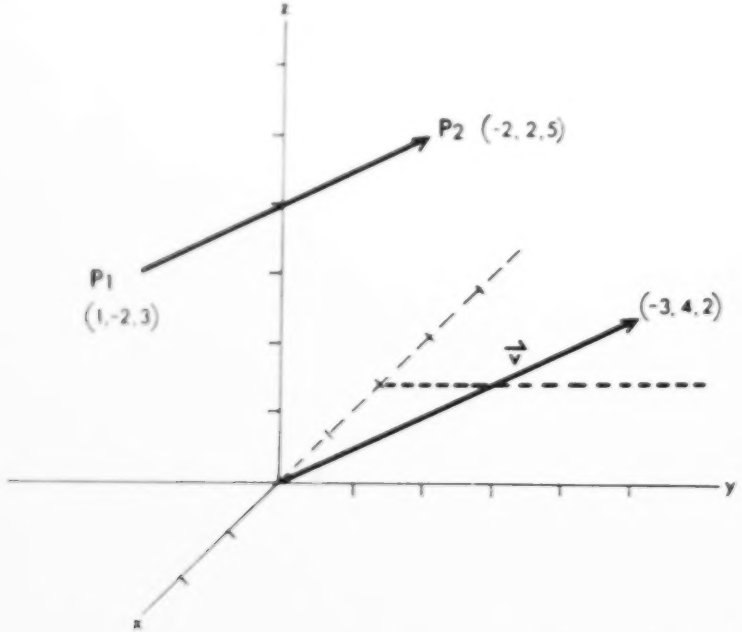
$$V = \vec{PQ} = 5\sqrt{2}$$

OBJECTIVES	NOTES TO TEACHERS

20. to represent any vector in space by its position vector;

Example:

Represent the vector $\overrightarrow{P_1P_2}$ determined by the point P (1,-2,3) and P (-2,2,5) by a position vector.



$\vec{v}$ is the position vector for $\overrightarrow{P_1P_2}$. See Objective 7.

OBJECTIVES	NOTES TO TEACHERS

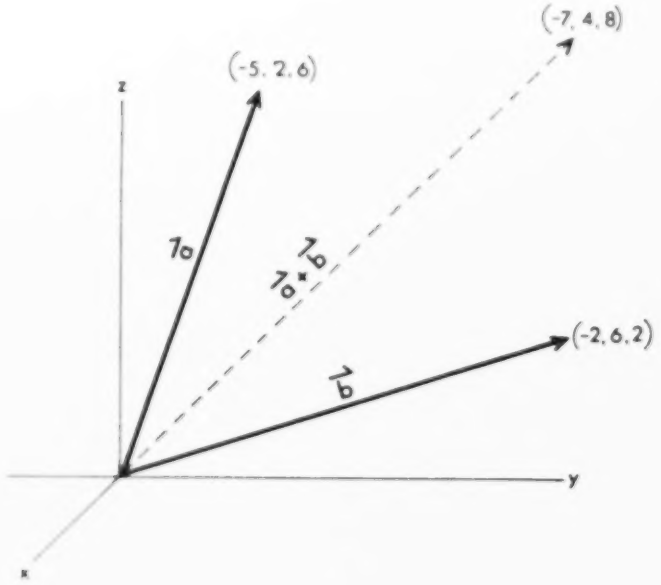
21. to add and subtract position vectors in 3-space by composition of coordinates;

Example:

If $\vec{a} = -5\vec{i} - 2\vec{j} + 6\vec{k}$ and

$\vec{b} = -2\vec{i} + 6\vec{j} + 2\vec{k}$

Find: $\vec{a} + \vec{b}$



(Do not draw boxes.)

22. to express any position vector in 3-space in terms of its components;

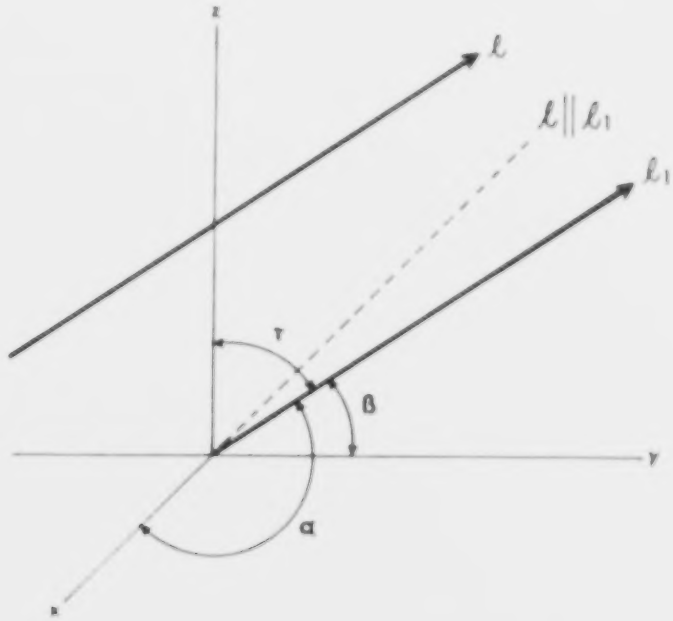
Example:

Resolve the vector $w = (3,-2,5)$ into its components which are collinear with e_1, e_2 and e_3.

Solution:

$w = (3,-2,5) = 3e_1 - 2e_2 + 5e_3$

$w_1 = 3e_1$, $w_2 = -2e_2$, $w_3 = 5e_3$

OBJECTIVES	NOTES TO TEACHERS
23. to add and subtract position vectors which have been expressed in terms of their components;	Examples: 1. Sketch the following vectors, add them and sketch their sum: $2\vec{i} + 3\vec{j}$, $-4\vec{j} + \vec{k}$ 2. Find a, b and c such that $5a\vec{i} + (b - 2)\vec{j} + 3\vec{k} = 10\vec{i} + 8\vec{j} - 3c\vec{k}$
24. to identify the direction angles of a vector in space;	

α, β and γ are the direction angles for $\vec{i}$.

OBJECTIVES	NOTES TO TEACHERS
25. to determine the direction cosines of a vector in space.	The formula for the direction cosines in space is analogous to the one used to determine the direction cosines for a line in a plane. If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are points on a directed line in space and d is the directed distance from P_1 to P_2, then the direction cosines are: $\text{Cos } \alpha = \frac{x_2 - x_1}{d}$ $\text{Cos } \beta = \frac{y_2 - y_1}{d}$ $\text{Cos } \gamma = \frac{z_2 - z_1}{d}$

appendices

APPENDIX I

ANNOTATIONS OF PRIMARY REFERENCES

Del Grande, et al. Calculus. Agincourt, ON: Gage Educational Publishers, 1979.

The text is well-written and illustrated with an excellent variety of documented examples. The approach is highly descriptive but incomplete. Topics are well-ordered and consistent. Graded questions are plentiful and answers to most questions are provided.

Edwards, R. Introduction to the Theory of Numbers. Boston, MA: Houghton Mifflin Publishers, 1975.

This pamphlet discusses the properties of integers by investigating divisibility, natural number functions and congruences. Theorems are developed very thoroughly and are suitably followed by selected exercises. Detailed solutions are provided for every exercise.

Elliott, H.A., et al. Calculus, Complex Numbers and Polar Co-ordinates. Toronto, ON: Holt, Rinehart and Winston, 1980.

This book is written at a reading level suitable for high-school students. Definitions and theorems are set out in colour. Adequate exercises graded in difficulty accompany each concept. Other features include chapter summaries, chapter reviews, mathematical tables and answers to most of the exercises.

Fisher, Robert C. and Ziebur, Allen D. Integrated Algebra and Trigonometry with Analytic Geometry. Third Edition. Toronto, ON: Prentice-Hall, 1982.

The material is developed thoroughly but concisely. Substantial problem sets follow the exposition of each topic. These are well graded and provide sufficient challenge for students. Symbolism is used in a simple and appropriate manner. The illustrations are limited in number but are readily comprehensible. Each chapter concludes with a set of review questions and a set of challenging problems. Mathematical tables and answers for most questions are provided.

Hart, W.L. Algebra, Elementary Functions and Probability. Toronto, ON: D.C. Heath Publications, 1965.

The reading level of this textbook is not suitable for the average high-school mathematics student. Some of the topics contained in this book are: complex numbers, theory of equations, mathematical induction and linear programming. There are numerous illustrations and many excellent exercises. Review and miscellaneous questions, a good table of contents, and selected answers are other good features of this book.

Shanks, Merrill E., et al. Pre-Calculus Mathematics. Second Edition. Don Mills, ON: Addison-Wesley Publishing, 1972.

Topics are developed concisely and are supported by an adequate number of graded exercises. Definitions and theorems are set out in colour, and many diagrams and graphs are interspersed throughout the text. Other features include: summaries and review questions at the end of each chapter, historical notes in most chapters, mathematical tables, lists of symbols and formulas, the Greek alphabet, and answers to selected problems.

A teacher's commentary which provides teaching suggestions and sample test items is available, as a detailed solutions manual.

APPENDIX II

ANNOTATIONS OF SUPPLEMENTARY REFERENCES

Ayres, F. First Year College Mathematics. Schaum's Outline Series. Scarborough, ON: McGraw-Hill Ryerson, 1958.

Schaum's outline is designed to supplement the course. Each topic is introduced by a brief presentation of material, followed by a collection of completely solved representative problems, and a section of supplementary problems for which answers but not solutions are given.

Included is material on systems of linear equations, complex numbers, theory of equations, calculus, polar co-ordinates and mathematical induction.

Friedberg, R. An Adventurer's Guide to Number Theory. Toronto, ON: McGraw-Hill Ryerson, 1975.

An elementary understanding of algebra and arithmetic is all that is necessary for the reader to understand and enjoy Friedberg's book. Numerous references to contributions made by mathematicians are sprinkled throughout the text. Prime number theory, divisibility properties and mathematical induction are discussed in an interesting manner.

Glecksman, A.M. An Introduction to Linear Programming and the Theory of Games. New York, NY: John Wiley and Sons, 1963.

This text introduces the student to linear programming and the theory of games in a strictly elementary yet mathematically sound manner. Detailed examples of the application of linear programming to problem solving are given. Other features include good diagrams, exercises, answers and easy readability.

Nichols, D., et al. Modern Intermediate Algebra. Revised Edition. Toronto, ON: Holt, Rinehart and Winston, 1969.

This text emphasizes reasoning and proof of algebraic concepts, and is designed to enable students to discover mathematical truths and patterns. An attempt is made to grade exercises according to degree of difficulty. Each chapter concludes with a vocabulary list, review exercise and test. A reading list, a glossary, an index and answers to selected exercises are contained within the text. Also available are a teacher's edition and a complete solution manual.

A marked decrease in the use of mathematical symbolism is evident in this edition as compared to the 1965 edition.

Protter, Murray H. and Morrey, Charles B. Calculus with Analytic Geometry: A First Course. Reading, MA: Addison-Wesley Publishing, 1977.

To make this text particularly attractive to students of science and engineering, considerable attention has been devoted to elementary topics, illustrative examples and intuitive explanations of many of the fundamental concepts. Physical applications are emphasized wherever suitable, there is a large selection of graded problems, definitions and theorems are stated clearly and the proofs of many theorems are given in full.

Vance, Elbraidge P. An Introduction to Modern Mathematics. Second Edition. Don Mills, ON: Addison-Wesley Publishing, 1968.

This edition presents a unified treatment of the fundamental ideas of Algebra, Trigonometry and Geometry. The author has included many graded problems, some of which serve to unify the material, while others introduce important concepts that are referred to in subsequent chapters. Review problems are found throughout the text and answers are provided for the odd numbered exercises. Mathematical tables are included.

An answer booklet for selected even-numbered problems is available.